1210 Midterm 2024

1. Use the principle of mathematical induction to show that for $n \geq 1$

$$1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

When $n = 1$:
L.S.$=1 \cdot 3 = 3$ R.S.$=\frac{1(2)(9)}{6} = 3$
Hence, the result is valid for $n = 1$. Assume that the result is valid for some integer $k \geq 1$; that is,

$$1 \cdot 3 + 2 \cdot 4 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}.$$

We must now prove that the result is valid for $k+1$; that is, we must prove that

$$1 \cdot 3 + 2 \cdot 4 + \cdots + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6}.$$

The left side is equal to

$$\begin{aligned}
[1 \cdot 3 + 2 \cdot 4 + \cdots + k(k+2)] + (k+1)(k+3) &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \\
&= \left(\frac{k+1}{6}\right)[k(2k+7) + 6(k+3)] \\
&= \left(\frac{k+1}{6}\right)(2k^2 + 13k + 18) \\
&= \frac{(k+2)(k+2)(2k+9)}{6}.
\end{aligned}$$

Since this is the right side of what we have to prove, the result is valid for $k+1$, and by the principle of mathematical induction, the result is valid for all $n \geq 1$.

2. Let $S_n = 1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2)$. Write S_n in sigma notation.

In sigma notation,

$$1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \sum_{k=1}^{n} k(k+2).$$

3. Compute $\dfrac{4-3i}{2+i} + \overline{3+4i} \cdot (2i)$.

$$\frac{4-3i}{2+i} + \overline{3+4i} \cdot (2i) = \frac{4-3i}{2+i} \cdot \frac{2-i}{2-i} + (3-4i)(2i) = \frac{5-10i}{5} + 8 + 6i = 9 + 4i$$

4. Compute $(\sqrt{3}-i)^{14}$. Write your answer in polar form using the principal value of the argument.

Since $\sqrt{3}-i = 2e^{-\pi i/6}$,

$$(\sqrt{3}-i)^{14} = 2^{14}e^{-14\pi i/6} = 2^{14}e^{-7\pi i/3} = 2^{14}e^{-\pi i/3} = 2^{14}\left[\cos\left(-\pi/3\right) + i\sin\left(-\pi/3\right)\right].$$

5. Let $P(x) = 8x^3 + 2x^2 + x + 3$.

(a) Using Descartes' rules of signs, determine the number of possible positive zeros and the number of negative zeros.

(b) Using the Bounds Theorem, determine the bound on the moduli ot the zeros.

(c) It is given that $P(-3/4) = 0$. Find all zeros of $P(x)$

(a) Since $P(x)$ has no sign changes, $P(x)$ has no positive zeros. Since $P(-x) = -8x^3 + 2x^2 - x + 3$ has 3 sign changes, $P(x)$ has either 3 or 1 negative zero.

(b) According to the Bounds Theorem, $|x| < \dfrac{3}{8} + 1 = \dfrac{11}{8}$.

(c) Since $x = -3/4$ is a zero, $4x + 3$ must be a factor of $P(x)$, and

$$P(x) = (4x+3)(2x^2 - x + 1).$$

The other two zeros are $x = \dfrac{1 \pm \sqrt{1-8}}{4} = \dfrac{1}{4} \pm \dfrac{\sqrt{7}}{4}i$.

6. Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \\ 4 & 2 \end{pmatrix}$. Compute the following if they are defined. If they are not, explain why not. Be specific.

(a) $\mathbf{AC} - 3\mathbf{B}$ (b) $(\mathbf{CA})\mathbf{B}$ (c) $\mathbf{CA} + \mathbf{I}_3$

(a) $\mathbf{AC} - 3\mathbf{B} = \begin{pmatrix} -7 & -10 \\ 24 & 13 \end{pmatrix} - 3\begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -16 & 5 \\ 21 & 7 \end{pmatrix}$

(b) $\mathbf{CA}$ is a 3×3 matrix. Since $\mathbf{B}$ is 2×2, the product $(\mathbf{CA})\mathbf{B}$ is not defined.

(c) $\mathbf{CA} + \mathbf{I}_3 = \begin{pmatrix} -3 & -8 & -9 \\ 8 & 5 & 10 \\ 8 & -2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -8 & -9 \\ 8 & 6 & 10 \\ 8 & -2 & 5 \end{pmatrix}$

7. For the points $P(\lambda, 2, -1)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$, where λ is a real number, either find all values of λ such $\angle PQR = \pi/3$ $(60°)$, or show that such λ do not exist.

Vector $\mathbf{QP} = \langle \lambda - 2, 1, 0 \rangle$, and $\mathbf{QR} = \langle -3, 0, 3 \rangle$. Since

$$\mathbf{QP} \cdot \mathbf{QR} = |\mathbf{QP}|\,|\mathbf{QR}| \cos \angle PQR,$$

it follows that

$$-3(\lambda - 2) = \sqrt{(\lambda-2)^2 + 1}\sqrt{18}\left(\frac{1}{2}\right) = \sqrt{(\lambda-2)^2+1}\left(\frac{3}{\sqrt{2}}\right).$$

When we divide by 3, and square both sides of the equation,

$$2(\lambda-2)^2 = (\lambda-2)^2 + 1 \qquad \Longrightarrow \qquad (\lambda-2)^2 = 1 \qquad \Longrightarrow \qquad \lambda = 1, 3.$$

But only $\lambda = 1$ satisfies the original equation in λ.

8. Suppose that we are given the point $P(0, 1, 2)$, the plane

$$\Pi : y + 2z - 1 = 0,$$

and the line

$$\ell : x = 1 - t, \quad y = -2, \quad z = 2 - 3t, \quad t \text{ in } \mathcal{R}.$$

(a) Find all points of intersection of line ℓ and plane Π, or show that such points do not exist.
(b) Find an equation of the plane Π_1 which contains the line ℓ and passes through the point P.

(a) When we substitute the parametric equations of the line into the equation of the plane,

$$-2 + 2(2 - 3t) - 1 = 0 \qquad \Longrightarrow \qquad -6t + 1 = 0 \qquad \Longrightarrow \qquad t = 1/6.$$

Hence the point of intersection is $(5/6, -2, 3/2)$.
(b) A vector along the line is $\langle -1, 0, -3 \rangle$. Since $Q(1, -2, 2)$ is a point on the line, the vector $\mathbf{PQ} = \langle 1, -3, 0 \rangle$ is in Π_1. Hence, a vector perpendicular to the plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 0 & -3 \\ 1 & -3 & 0 \end{vmatrix} = \langle -9, -3, 3 \rangle,$$

as is $\langle 3, 1, -1 \rangle$. The equation for Π_1 is therefore $3(x) + 1(y-1) - (z-2) = 0$, or, $3x + y - z = -1$.