1210 Assignment 3 Solutions

1. Consider the matrix

$$A = \begin{pmatrix} 3 & -1 & 4 & 7 \\ 1 & 2 & -5 & 8 \end{pmatrix}.$$

(a) Find the reduced row-echelon form of A by first dividing the first row by 3. Now use the 1 in the $(1,1)$ position to make the $(2,1)$ entry zero and so on. Show all of your row operations carefully.

(b) Now, starting from A again, find the reduced row-echelon form of A as follows: Switch the two rows, and then use the 1 in the $(1,1)$ position to make the $(2,1)$ entry zero and so on.

(a)

$$\begin{pmatrix} 3 & -1 & 4 & 7 \\ 1 & 2 & -5 & 8 \end{pmatrix} R_1 \to R_1/3$$

$$\longrightarrow \begin{pmatrix} 1 & -1/3 & 4/3 & 7/3 \\ 1 & 2 & -5 & 8 \end{pmatrix} R_2 \to -R_1 + R_2$$

$$\longrightarrow \begin{pmatrix} 1 & -1/3 & 4/3 & 7/3 \\ 0 & 7/3 & -19/3 & 17/3 \end{pmatrix} R_2 \to 3R_2/7$$

$$\longrightarrow \begin{pmatrix} 1 & -1/3 & 4/3 & 7/3 \\ 0 & 1 & -19/7 & 17/7 \end{pmatrix} R_1 \to R_2/3 + R_1$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3/7 & 22/7 \\ 0 & 1 & -19/7 & 17/7 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 3 & -1 & 4 & 7 \\ 1 & 2 & -5 & 8 \end{pmatrix} \begin{matrix} R_1 \to R_2 \\ R_2 \to R_1 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & -5 & 8 \\ 3 & -1 & 4 & 7 \end{pmatrix} R_2 \to -3R_1 + R_2$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & -5 & 8 \\ 0 & -7 & 19 & -17 \end{pmatrix} R_2 \to -R_2/7$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & -5 & 8 \\ 0 & 1 & -19/7 & 17/7 \end{pmatrix} R_1 \to -2R_2 + R_1$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3/7 & 22/7 \\ 0 & 1 & -19/7 & 17/7 \end{pmatrix}$$

2. Consider the three vectors

$$\mathbf{u} = \langle a+3, 1, 1\rangle, \qquad \mathbf{v} = \langle 1, a+2, 2\rangle, \qquad \mathbf{w} = \langle 1, 2, a+2\rangle,$$

where a is a real number.
(a) Find all values of a such that the vectors are linearly dependent.
(b) For each of the values of a found in part (a), express $\mathbf{v}$ as a linear combination of the vectors $\mathbf{u}$ and $\mathbf{w}$.

(a) The vectors are linearly dependent if

$$0 = \begin{vmatrix} a+3 & 1 & 1 \\ 1 & a+2 & 2 \\ 1 & 2 & a+2 \end{vmatrix} = (a+3)[(a+2)^2 - 4] - (a+2-2) + (2-a-2)$$
$$= (a+3)(a^2+4a) - 2a = a^3 + 7a^2 + 10a = a(a+2)(a+5).$$

Thus, $a = 0,\ -2,\ -5$.
(b) When $a = 0$, the vectors are

$$\mathbf{u} = \langle 3, 1, 1\rangle, \qquad \mathbf{v} = \langle 1, 2, 2\rangle, \qquad \mathbf{w} = \langle 1, 2, 2\rangle.$$

Clearly, $\mathbf{v} = \mathbf{w}$.
When $a = -2$, the vectors are

$$\mathbf{u} = \langle 1, 1, 1\rangle, \qquad \mathbf{v} = \langle 1, 0, 2\rangle, \qquad \mathbf{w} = \langle 1, 2, 0\rangle.$$

We see that $\mathbf{v} = 2\mathbf{u} - \mathbf{w}$. If this is not evident, we consider finding constants C_1 and C_2 such that

$$\mathbf{v} = C_1\mathbf{u} + C_2\mathbf{w} \quad \Longrightarrow \quad \langle 1, 0, 2\rangle = C_1\langle 1, 1, 1\rangle + C_2\langle 1, 2, 0\rangle.$$

When we equate components,

$$1 = C_1 + C_2, \quad 0 = C_1 + 2C_2, \quad 2 = C_1.$$

The solution is $C_1 = 2$, $C_2 = -1$ and therefore $\mathbf{v} = 2\mathbf{u} - \mathbf{w}$.
When $a = -5$, the vectors are

$$\mathbf{u} = \langle -2, 1, 1\rangle, \qquad \mathbf{v} = \langle 1, -3, 2\rangle, \qquad \mathbf{w} = \langle 1, 2, -3\rangle.$$

We can see that $\mathbf{v} = -\mathbf{u} - \mathbf{w}$. If this is not evident, we can apply the same procedure as we did for $a = -2$.

3. It is given to you that the matrix

$$A = \begin{bmatrix} a+b-3 & b+3a & 0 & 1 \\ 0 & 0 & 1 & 4a-7b \\ 0 & 0 & 3a+2b-5 & 0 \end{bmatrix}$$

is in reduced row-echelon form. Find the values of a and b.

One possibility for RREF is to have the $(1,1)$ entry equal to 1 and the $(3,3)$ entry equal to 0, in which case

$$a+b-3=1, \quad 3a+2b-5=0, \qquad \text{or} \qquad a+b=4, \quad 3a+2b=5.$$

By Cramer's rule,

$$a = \frac{\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{3}{-1} = -3, \qquad b = \frac{\begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix}}{-1} = \frac{-7}{-1} = 7.$$

A second possibility is to have the $(1,1)$ entry equal to 0, the $(1,2)$ entry equal to 1, and the $(3,3)$ entry equal to 0, in which case

$$a+b-3=0, \quad b+3a=1, \quad 3a+2b-5=0,$$

or,

$$a+b=3, \quad 3a+b=1, \quad 3a+2b=5.$$

We use augmented matrices to see if this system of three equations in two unknowns has a solution,

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & 1 & 1 \\ 3 & 2 & 5 \end{array}\right) \begin{array}{l} R_2 \to -3R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{array} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -8 \\ 0 & -1 & -4 \end{array}\right) R_2 \to -R_2/2$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \end{array}\right) \begin{array}{l} R_1 \to -R_2 + R_1 \\ \\ R_3 \to R_2 + R_3 \end{array} \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

Thus, $a = -1$ and $b = 4$.

4. Consider the following system of equations:

$$x + y + z = 1, \qquad -2x + y - 3z = 0, \qquad 5x - 8y + 11z = 3.$$

(a) Solve the system using Gauss-Jordan elimination.
(b) Now solve the same system using Cramer's rule.

(a)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -2 & 1 & -3 & 0 \\ 5 & -8 & 11 & 3 \end{array}\right) \begin{array}{l} \\ R_2 \to 2R_1 + R_2 \\ R_3 \to -5R_1 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & -13 & 6 & -2 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to 4R_2 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & -1 & 2 & 6 \end{array}\right) \begin{array}{l} \\ R_2 \to -R_3 \\ R_3 \to R_2 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -6 \\ 0 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \to -R_2 + R_1 \\ \\ R_3 \to -3R_2 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 5 & 20 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to R_3/5 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 4 \end{array}\right) \begin{array}{l} R_1 \to -3R_3 + R_1 \\ R_2 \to 2R_3 + R_2 \\ \\ \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

Hence, the solution is $x = -5$, $y = 2$, and $z = 4$.
(b) By Cramer's rule,

$$x = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 3 & -8 & 11 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & -3 \\ 5 & -8 & 11 \end{vmatrix}} = \frac{1(-13) + 3(-4)}{1(-13) + 2(19) + 5(-4)} = -5,$$

$$y = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & -3 \\ 5 & 3 & 11 \end{vmatrix}}{5} = \frac{-1(-7) - 3(-1)}{5} = 2,$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 5 & -8 & 3 \end{vmatrix}}{5} = \frac{1(11)+3(3)}{5} = 4.$$

5. Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Find $\det(M)$. Hint: Use row operations to convert the determinant into a form which can be calculated more easily.

If we add -1 times the first row to rows 2, 3, 4, and 5, we get

$$\det(M) = \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

If we expand along the first column,

$$\det(M) = \det \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now, add rows 1, 2, 3, and 4 to row 5,

$$\det(M) = \det \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Now expand along the last row,

$$\det(M) = 5 \det \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = 5.$$