Midterm Test October 22, 2018

20 **1.** Use mathematical induction to prove that

$$n + (n+1) + (n+2) + \cdots + (2n) = \frac{3n(n+1)}{2}, \qquad \text{for } n \geq 1.$$

When $n = 1$, the left side is $1 + 2 = 3$, and the right side is $3(1)(2)/2 = 3$. Thus, the result is valid for $n = 1$. Assume that it is valid for some integer k; that is,

$$k + (k+1) + (k+2) + \cdots + (2k) = \frac{3k(k+1)}{2}.$$

We now want to prove that the result is valid for $k+1$; that is,

$$(k+1) + (k+2) + (k+3) + \cdots + (2k+2) = \frac{3(k+1)(k+2)}{2}.$$

The left side is equal to

$$\begin{aligned} \text{L.S.} &= [k + (k+1) + (k+2) + \cdots + (2k)] + (2k+1) + (2k+2) - k \\ &= \frac{3k(k+1)}{2} + 3k + 3 = 3(k+1)\left(\frac{k}{2} + 1\right) = \frac{3(k+1)(k+2)}{2}. \end{aligned}$$

Thus, the result is valid for $k+1$, and by mathematical induction, it is valid for all $n \geq 1$.

10 **2.** Evaluate the summation

$$\sum_{i=3}^{39} [(2i-1)^2 + 4i].$$

Simplify your answer. You may use any of the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

$$\begin{aligned} \sum_{i=3}^{39} [(2i-1)^2 + 4i] &= \sum_{i=3}^{39} [4i^2 + 1] = \sum_{i=1}^{39} (4i^2 + 1) - 5 - 17 \\ &= 4\sum_{i=1}^{39} i^2 + 39 - 22 = 4\left[\frac{39(40)(79)}{6}\right] + 17 = 82,177. \end{aligned}$$

6 **3.** Find the imaginary part of the complex number $\dfrac{(\overline{3+2i})(1+i)^2}{2-i}$.

$$\frac{(\overline{3+2i})(1+i)^2}{2-i} = \frac{(3-2i)(2i)}{2-i} = \frac{4+6i}{2-i}\frac{2+i}{2+i} = \frac{2+16i}{5}$$

The imaginary part is 16/5.

14 **4.** Consider the polynomial equation

$$P(x) = 3x^5 - 2x^4 - 3x^3 + 2x^2 + x - 16 = 0.$$

(a) What do Descartes' rules of sign predict for the numbers of positive and negative solutions to the equation?

(b) What does the bounds theorem say about solutions to the equation?

(c) Taking parts (a) and (b) into account, what are the possible rational roots of the equation as predicted by the rational root theorem?

(a) Since $P(x)$ has 3 signs changes, there are 3 or 1 positive roots. Since

$$P(-x) = -3x^5 - 2x^4 + 3x^3 + 2x^2 - x - 16$$

has 2 signs changes, there is 2 or 0 negative roots.

(b) Since $M = 16$, the bounds theorem indicates that $|x| < \dfrac{16}{3} + 1 = \dfrac{19}{3}$.

(c) Possible rational roots are

$$\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{16}{3}.$$

6 **5.** If matrices **A** and **B** are

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 4 \\ 2 & 1 & 5 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 6 & 1 & 0 \\ 2 & 0 & 3 \\ -4 & 1 & 5 \end{pmatrix},$$

and $\mathbf{C} = 3\mathbf{A} - \mathbf{A}\mathbf{B}^T$, find the entry c_{32} of $\mathbf{C}$.

Since the $(3,2)$ entry of

$$\mathbf{A}\mathbf{B}^T = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 4 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 6 & 2 & -4 \\ 1 & 0 & 1 \\ 0 & 3 & 5 \end{pmatrix}$$

is 19, the $(3,2)$ entry of $\mathbf{C}$ is $3(1) - 19 = -16$.

12 **6.** Find the equation of the plane that contains the point $(4, 2, -3)$ and the line

$$L: \quad x = 2 + t, \ y = 4 - t, \ z = 3 + 2t.$$

A vector along the line is $\langle 1, -1, 2\rangle$. Since $(2, 4, 3)$ is a point on the line, a second vector in the plane is $(4, 2, -3) - (2, 4, 3) = \langle 2, -2, -6\rangle$. A vector normal to the plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 2 & -2 & -6 \end{vmatrix} = \langle 10, 10, 0\rangle,$$

and so also is $\langle 1, 1, 0\rangle$. The equation of the plane is

$$x + y = D.$$

Since $(2, 4, 3)$ is a point on the plane $2 + 4 = D$, and therefore the equation of the plane is

$$x + y = 6.$$