Midterm Test October 23, 2018

20 **1.** Use mathematical induction to prove that 576 divides $5^{2n+2} - 24n - 25$ for $n \geq 1$.

When $n = 1$, $5^{2n+2} - 24n - 25 = 5^4 - 24 - 25 = 576$, which is certainly divisible by 576. Assume that 576 divides

$$5^{2k+2} - 24k - 25.$$

Consider

$$\begin{aligned} 5^{2k+4} - 24(k+1) - 25 = 25(5^{2k+2}) - 24k - 49 &= 25(5^{2k+2} - 24k - 25) - 24k - 49 - 25(-24k - 25) \\ &= 25(5^{2k+2} - 24k - 25) + 576k + 576. \end{aligned}$$

Since 576 divides each term on the right, 576 divides the expression on the left; that is, the result is valid for $k + 1$, and hence by mathematical induction, it is valid for all $n \geq 1$.

10 **2.** Evaluate the summation

$$\sum_{i=25}^{96} (4i - 5).$$

Simplify your answer. You may use any of the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\begin{aligned} \sum_{i=25}^{96} (4i - 5) &= \sum_{i=1}^{96} (4i - 5) - \sum_{i=1}^{24} (4i - 5) \\ &= 4\sum_{i=1}^{96} i - 5(96) - 4\sum_{i=1}^{24} i + 5(24) \\ &= 4\left[\frac{96(97)}{2}\right] - 360 - 4\left[\frac{24(25)}{2}\right] \\ &= 17,064 \end{aligned}$$

6 **3.** Find the imaginary part of the complex number $\dfrac{(2+4i)i^{15}\overline{(3-i)}}{1+i}$.

$$\begin{aligned}\frac{(2+4i)i^{15}\overline{(3-i)}}{1+i} &= \frac{(2+4i)(-i)(3+i)}{1+i} = \frac{(2+4i)(1-3i)}{1+i} \\ &= \frac{14-2i}{1+i}\frac{1-i}{1-i} = \frac{12-16i}{2} = 6-8i\end{aligned}$$

The imaginary part is -8.

14 **4.** Consider the polynomial equation

$$P(x) = 2x^6 + 4x^4 - 3x^2 + 2x - 15 = 0.$$

(a) What do Descartes' rules of sign predict for the numbers of positive and negative solutions to the equation?

(b) What does the bounds theorem say about solutions to the equation?

(c) Taking parts (a) and (b) into account, what are the possible rational roots of the equation as predicted by the rational root theorem?

(a) Since $P(x)$ has 3 signs changes, there are 3 or 1 positive roots. Since

$$P(-x) = 2x^6 + 4x^4 - 3x^2 - 2x - 15$$

has 1 sign change, there is 1 negative root.

(b) Since $M = 15$, the bounds theorem indicates that $|x| < \dfrac{15}{2} + 1 = \dfrac{17}{2}$.

(c) Possible rational roots are

$$\pm 1, \pm 3, \pm 5, \pm\frac{1}{2}, \pm\frac{3}{2} \pm \frac{5}{2}, \pm\frac{15}{2}.$$

6 **5.** If matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 6 & 1 & 0 \\ 2 & 3 & -4 \\ 0 & 5 & -2 \end{pmatrix},$$

and $\mathbf{C} = 2\mathbf{A} - \mathbf{A}^T\mathbf{B}$, find the entry c_{13} of $\mathbf{C}$.

Since the $(1, 3)$ entry of

$$\mathbf{A}^T\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ -3 & 4 & 3 \end{pmatrix} \begin{pmatrix} 6 & 1 & 0 \\ 2 & 3 & -4 \\ 0 & 5 & -2 \end{pmatrix}$$

is -4, the $(1, 3)$ entry of $\mathbf{C}$ is $2(-3) + 4 = -2$.

12 **6.** Show that the following lines determine a plane and find its equation.

$$L_1 : \quad x = 1 + 2t,\ y = 3 - 4t,\ z = -2 + 5t : \qquad L_2; \quad x = 1 - 4t,\ y = 3 + 2t,\ z = -2 + 3t.$$

Clearly, the lines intersect at the point $(1, 3, -2)$, and therefore make a plane. Since vectors along the lines are $\langle 2, -4, 5\rangle$ and $\langle -4, 2, 3\rangle$, a vector normal to the plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -4 & 2 & 3 \\ 2 & -4 & 5 \end{vmatrix} = \langle 22, 26, 12\rangle,$$

as is $\langle 11, 13, 6\rangle$. The equation of the plane is

$$11x + 13y + 6z = D.$$

Since $(1, 3, -2)$ is on the plane, $11(1) + 13(2) + 6(-2) = D$, so that $D = 38$. The equation of the plane is therfore

$$11x + 13y + 6z = 38.$$