Fall 2020 Exam Solutions

10 **1.** Find the interval of convergence for the series

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{(n+3)3^n}(2x-4)^n.$$

First we write the series in the form $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n 2^n}{(n+3)3^n}(x-2)^n$

The radius of convergence of the series is

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^n 2^n}{(n+3)3^n}}{\dfrac{(-1)^{n+1}2^{n+1}}{(n+4)3^{n+1}}} \right| = \frac{3}{2}.$$

The open interval of convergence is

$$|x-2| < \frac{3}{2} \quad \Longrightarrow \quad -\frac{3}{2} < x-2 < \frac{3}{2} \quad \Longrightarrow \quad \frac{1}{2} < x < \frac{7}{2}.$$

At $x = 1/2$, the series becomes $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n 2^n}{(n+3)3^n}\left(-\frac{3}{2}\right)^n = \sum_{n=4}^{\infty} \frac{1}{n+3}$.

Since this is the harmonic series (less the first six terms), the series diverges.

At $x = 7/2$, the series becomes $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n 2^n}{(n+3)3^n}\left(\frac{3}{2}\right)^n = \sum_{n=4}^{\infty} \frac{(-1)^n}{n+3}$.

Since this is the alternating harmonic series (less the first six terms), the series converges. The interval of convergence is therefore $-1/2 < x \leq 7/2$.

12 **2.** Find the Taylor series about $x = 1$ for the function

$$f(x) = \frac{x-1}{(3x+1)^{1/3}}.$$

Write your final answer in sigma notation, simplified as much as possible. You must use a method that guarantees that the series converges to $f(x)$. What is the radius of convergence of the series?

$\boxed{x-1}$

$$\begin{aligned}
\frac{1}{(3x+1)^{1/3}} &= \frac{1}{[3(x-1)+4]^{1/3}} = \frac{1}{4^{1/3}}\left[1+\frac{3(x-1)}{4}\right]^{-1/3} \\
&= \frac{1}{4^{1/3}}\left\{1+(-1/3)\left[\frac{3}{4}(x-1)\right]+\frac{(-1/3)(-4/3)}{2!}\left[\frac{3}{4}(x-1)\right]^2\right. \\
&\qquad \left. + \frac{(-1/3)(-4/3)(-7/3)}{3!}\left[\frac{3}{4}(x-1)\right]^3+\cdots\right\} \\
&= \frac{1}{4^{1/3}}\left[1-\frac{1}{4}(x-1)+\frac{(1\cdot 4)}{2!\,4^2}(x-1)^2-\frac{(1\cdot 4\cdot 7)}{3!\,4^3}(x-1)^3+\cdots\right] \\
&= \frac{1}{4^{1/3}}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 4\cdot 7\cdots(3n-2)]}{n!\,4^n}(x-1)^n\right\} \\
&= \frac{1}{4^{1/3}}+\sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 4\cdot 7\cdots(3n-2)]}{n!\,4^{n+1/3}}(x-1)^n
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{x-1}{(3x+1)^{1/3}} &= \frac{1}{4^{1/3}}(x-1)+\sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 4\cdot 7\cdots(3n-2)]}{n!\,4^{n+1/3}}(x-1)^{n+1} \\
&= \frac{1}{4^{1/3}}(x-1)+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}[1\cdot 4\cdot 7\cdots(3n-5)]}{(n-1)!\,4^{n-2/3}}(x-1)^n.
\end{aligned}$$

Since the series is valid for $\left|\frac{3}{4}(x-2)\right| < 1 \implies |x-2| < 4/3$, the radius of converence is $4/3$.

8 **3.** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{3^{2n+2}}(3x-4)^{n+1}.$$

You need **NOT** simplify your answer. Include the interval of convergence.

When we write the series in the form

$$\sum_{n=1}^{\infty} \frac{(3x-4)}{9}\left[-\frac{(3x-4)}{9}\right]^n,$$

we see that it is geometric with sum

$$\frac{\dfrac{(3x-4)}{9}\left[-\dfrac{(3x-4)}{9}\right]}{1+\dfrac{3x-4}{9}}.$$

The interval of convergence is

$$\left|\frac{-(3x-4)}{9}\right| < 1 \implies |3x-4| < 9 \implies -9 < 3x-4 < 9 \implies -5 < 3x < 13 \implies -\frac{5}{3} < x < \frac{13}{3}.$$

6 **4.** Find a one-parameter family of solutions for the differential equation

$$y\frac{dy}{dx} = (y-1)x.$$

Does your family have any singular solutions? Explain.

We can separate the equation

$$\frac{y}{y-1}dy = x\,dx \qquad \text{provided } y \neq 1.$$

A 1-parameter family of solutions is defined implicitly by

$$\int\left(1+\frac{1}{y-1}\right)dy = \frac{x^2}{2}+C \quad \implies \quad y+\ln|y-1| = \frac{x^2}{2}+C.$$

Since $y=1$ is a solution of the differential equation, but it is not within the family, it is a singular solution of the family.

6 **5.** You are given that the roots of the auxiliary equation associated with the linear differential equation

$$\phi(D)y = 3x^2+4+2\cos 4x - e^x \sin x$$

are $m = 1, 2\pm i, 0, 0, 0, 1\pm\sqrt{2}, \pm 4i, \pm 4i$. What is the form for a particular solution of the differential equation as predicted by the method of undetermined coefficients?

$$\begin{aligned} y_h(x) = {} & C_1e^x + e^{2x}(C_2\cos x + C_3\sin x) + C_4 + C_5x + C_6x^2 + C_7e^{(1+\sqrt{2})x} + C_8e^{(1-\sqrt{2})x} \\ & + (C_9 + C_{10}x)\cos 4x + (C_{11}+C_{12}x)\sin 4x \end{aligned}$$

$$y_p(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2\cos 4x + Ex^2\sin 4x + Fe^x\sin x + Ge^x\cos x$$

12 **6.** A container of water at temperature 90° C is taken outside where the temperature is $-10°$ C, but it is falling at the rate of 1 degree per hour. Newton's law of cooling states that the rate of change of the temperature of the water is proportional to the difference between the temperature of the water and the outside temperature. Find the temperature of the water for the first three hours.

Since the outside temperature is $-10 - t$, temperature $T(t)$ of the water (in hours) must satisfy

$$\frac{dT}{dt} = k(T + 10 + t), \qquad T(0) = 90.$$

This equation is linear first-order, with integrating factor e^{-kt}, and therefore we write

$$e^{-kt}\frac{dT}{dt} - ke^{-kt}T = k(10 + t)e^{-kt} \qquad \Longrightarrow \qquad \frac{d}{dt}(Te^{-kt}) = k(10 + t)e^{-kt}.$$

Integration gives

$$Te^{-kt} = \int k(10 + t)e^{-kt}\,dt + C = -\left(10 + \frac{1}{k} + t\right)e^{-kt} + C.$$

Thus,

$$T(t) = -\left(10 + \frac{1}{k} + t\right) + Ce^{kt}.$$

The initial condition requires $90 = -\left(10 + \frac{1}{k}\right) + C$. Finally, then

$$T(t) = -\left(10 + \frac{1}{k} + t\right) + \left(100 + \frac{1}{k}\right)e^{kt}.$$

6 **7.** The differential equation associated with a vibrating mass-spring system is

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 16x = A\sin\omega t.$$

You are told that a general solution of the differential equation is

$$x(t) = e^{-t/2}\left(C_1\cos\frac{\sqrt{63}t}{2} + C_2\sin\frac{\sqrt{63}t}{2}\right) + \frac{A(16-\omega^2)}{(16-\omega^2)^2+\omega^2}\sin\omega t - \frac{\omega A}{(16-\omega^2)^2+\omega^2}\cos\omega t.$$

DO NOT DERIVE THIS SOLUTION; ACCEPT IT.

Answer the following questions:

(a) If the differential equation describes motion of a mass suspended from a spring, set up, **but do not solve**, equations that define C_1 and C_2 if initially the mass is raised 12 centimetres above its equilibrium position and given velocity 3 metres per second downward.

(b) What is the transient part of the solution?

(c) What is the steady-state part of the solution?

(d) What is the amplitude of the steady-state part of the solution. You need not simplify your answer.

(e) Is there a value of ω that will lead to unbounded oscillations?

(a) $\frac{12}{100} = C_1 - \frac{\omega A}{(16-\omega^2)^2+\omega^2}$, $\qquad -3 = -\frac{C_1}{2} + \frac{\sqrt{63}}{2}C_2 + \frac{A\omega(16-\omega^2)}{(16-\omega^2)^2+\omega^2}$

(b) $e^{-t/2}\left(C_1\cos\frac{\sqrt{63}t}{2} + C_2\sin\frac{\sqrt{63}t}{2}\right)$

(c) $\frac{A(16-\omega^2)}{(16-\omega^2)^2+\omega^2}\sin\omega t - \frac{\omega A}{(16-\omega^2)^2+\omega^2}\cos\omega t$

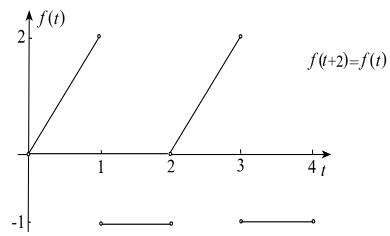
(d) $\sqrt{\left[\frac{A(16-\omega^2)}{(16-\omega^2)^2+\omega^2}\right]^2 + \left[-\frac{\omega A}{(16-\omega^2)^2+\omega^2}\right]^2}$

(e) No

14 **8.** Find Laplace transforms for the following functions:

(a) $f(t) = t^2 e^{-2t} h(t-3)$

(b)



(a)

$$F(s) = e^{-3s}\mathcal{L}\left\{(t+3)^2 e^{-2(t+3)}\right\} = e^{-3s}e^{-6}\mathcal{L}\left\{e^{-2t}(t^2+6t+9)\right\}$$
$$= e^{-3(s+2)}\mathcal{L}\left\{t^2+6t+9\right\}_{|s\to s+2} = e^{-3(s+2)}\left[\frac{2}{(s+2)^3} + \frac{6}{(s+2)^2} + \frac{9}{s+2}\right]$$

(b)

$$\begin{aligned}
F(s) &= \frac{1}{1-e^{-2s}}\mathcal{L}\left\{2t[h(t)-h(t-1)] - [h(t-1)-h(t-2)]\right\} \\
&= \frac{1}{1-e^{-2s}}\mathcal{L}\left\{2t - (2t+1)h(t-1) + h(t-2)\right\} \\
&= \frac{1}{1-e^{-2s}}\left[\frac{2}{s^2} - e^{-s}\mathcal{L}\left\{2(t+1)+1\right\} + \frac{e^{-2s}}{s}\right] \\
&= \frac{1}{1-e^{-2s}}\left[\frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2}+\frac{3}{s}\right) + \frac{e^{-2s}}{s}\right]
\end{aligned}$$

14 **9.** Find inverse Laplace transforms for the following functions:

$$\text{(a)}\quad F(s) = \frac{s^2+s+4}{(s^2+4)^2} \qquad\qquad \text{(b)}\quad F(s) = \frac{s+1}{(s^3-4s)(1+e^{-2s})}$$

(a)

$$f(t) = \mathcal{L}^{-1}\left\{\frac{(s^2+4)+s}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4} + \frac{s}{(s^2+4)^2}\right\} = \frac{1}{2}\sin 2t + \frac{t}{4}\sin 2t$$

(b)

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-2)(s+2)(1+e^{-2s})}\right\} \\ &= \mathcal{L}^{-1}\left\{\left[\frac{-1/4}{s} + \frac{3/8}{s-2} - \frac{1/8}{s+2}\right]\sum_{n=0}^{\infty}(-e^{-2s})^n\right\} \\ &= \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\left(-\frac{1/4}{s} + \frac{3/8}{s-2} - \frac{1/8}{s+2}\right)(-1)^n e^{-2ns}\right\} \\ &= \sum_{n=0}^{\infty}(-1)^n\left[-\frac{1}{4} + \frac{3}{8}e^{2(t-2n)} - \frac{1}{8}e^{-2(t-2n)}\right]h(t-2n) \end{aligned}$$

12 **10.** A 1 kilogram mass hangs from a spring with constant 10 newtons per metre. At time $t = 0$, it is given velocity 2 metres per second upward. During its motion, it is acted upon by a damping force equal to four times its velocity. At the 10 second mark, it is hit with a hammer that imparts 5 units of momentum downward to the mass. Find the position of the mass as a function of time. What is the change in velocity of the mass as a result of the hit?

Displacement $x(t)$ of the mass from equilibrium satisfies

$$1\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 10x = -5\delta(t-10), \qquad x(0) = 0, \quad x'(0) = 2.$$

When we take Laplace transforms,

$$(s^2X - 2) + 4(sX) + 10X = -5e^{-10s}.$$

Solving for $X(s)$ gives

$$X(s) = \frac{2-5e^{-10s}}{s^2+4s+10} = \frac{2}{(s+2)^2+6} - \frac{5e^{-10(s+2)}e^{20}}{(s+2)^2+6}.$$

Inverse transforms give

$$\begin{aligned} x(t) &= 2e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+6}\right\} - 5e^{-2t}e^{20}\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s^2+6}\right\} \\ &= \frac{2}{\sqrt{6}}e^{-2t}\sin\sqrt{6}t - \frac{5}{\sqrt{6}}e^{-2(t-10)}\sin\sqrt{6}(t-10)h(t-10) \text{ m.} \end{aligned}$$

The change in velocity is -5 metres per second.