Solutions to fall 2021 final exam

1. Find the sum of the following series, and determine the values of x for which the sum is valid. It is not necessary to simplify your expression for the sum.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+2}}{3^{2n+1}} x^{2n+1/2}$$

If we remove the square root from the series, the remaining series becomes clearly geometric.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+2}}{3^{2n+1}} x^{2n+1/2} = \sqrt{x} \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+2}}{3^{2n+1}} x^{2n} = \sqrt{x} \sum_{n=1}^{\infty} \frac{4}{3} \left(-\frac{2x^2}{9}\right)^n$$
$$= \sqrt{x} \frac{(4/3)(-2x^2/9)}{1 + 2x^2/9}$$

The geometric sum is valid for

$$\left|\frac{-2x^2}{9}\right| < 1 \quad \Longrightarrow \quad |x^2| < \frac{9}{2} \quad \Longrightarrow \quad -\frac{3}{\sqrt{2}} < x < \frac{3}{\sqrt{2}}.$$

But because of the square root, we must choose $0 \le x < 3/\sqrt{2}$.

2. Find the Taylor series about $x = 5$ for the function

$$f(x) = (x - 5) \ln(2x + 3).$$

Write your final answer in sigma notation, simplified as much as possible. You must use a method that guarantees that the series converges to $f(x)$. Determine the open interval of convergence of the series.

$\boxed{x - 5}$ We begin with

$$\frac{1}{2x+3} = \frac{1}{2(x-5)+13} = \frac{1/13}{1 + \frac{2}{13}(x-5)} = \frac{1}{13} \sum_{n=0}^{\infty} \left[-\frac{2}{13}(x-5)\right]^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{13^{n+1}} (x-5)^n,$$

valid for

$$\left|-\frac{2}{13}(x-5)\right| < 1 \quad \Longrightarrow \quad |x-5| < \frac{13}{2} \quad \Longrightarrow \quad -\frac{13}{2} < x - 5 < \frac{13}{2} \quad \Longrightarrow \quad -\frac{3}{2} < x < \frac{23}{2}.$$

Because the series has a positive radius of convergence, we may integrate

$$\frac{1}{2} \ln(2x+3) + C = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n+1) 13^{n+1}} (x-5)^{n+1}.$$

When we substitute $x = 5$,

$$\frac{1}{2} \ln 13 + C = 0 \quad \Longrightarrow \quad C = -\frac{1}{2} \ln 13.$$

Thus,

$$\frac{1}{2}\ln(2x+3) = \frac{1}{2}\ln 13 + \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(n+1)13^{n+1}}(x-5)^{n+1}.$$

Multiplication by $2(x-5)$ gives

$$\begin{aligned}(x-5)\ln(2x+3) &= \ln 13(x-5) + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+1}}{(n+1)13^{n+1}}(x-5)^{n+2}\\ &= \ln 13(x-5) + \sum_{n=2}^{\infty}\frac{(-1)^n 2^{n-1}}{(n-1)13^{n-1}}(x-5)^{n}.\end{aligned}$$

The open interval of convergence is $-3/2 < x < 23/2$.

3. You are given that the Maclaurin series for a function $f(x)$ is

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{2^n}x^n.$$

Find the 2021^{th} derivative of the function at $x=0$.

The coefficient of x^{2021} in the Maclaurin series for $f(x)$ is $\dfrac{f^{(2021)}(0)}{2021!}$. Thus

$$\frac{f^{(2021)}(0)}{2021!} = \frac{(-1)^{2021}(2021)^2}{2^{2021}} \qquad\Longrightarrow\qquad f^{(2021}(0) = -\frac{(2021)^2(2021)!}{2^{2021}}.$$

4. Show that the initial-value problem

$$\frac{dy}{dx} = \frac{y + x^2\cos x}{x}, \quad y(0)=0,$$

has infinitely many solutions.

Because the differential equation is linear first-order, $\dfrac{dy}{dx} - \dfrac{y}{x} = x\cos x$, an integrating factor is $e^{\int(-1/x)dx} = e^{-\ln|x|} = \dfrac{1}{|x|}$. If we multiply the differential equation by $1/x$, or by $-1/x$, the result is

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \cos x \qquad\Longrightarrow\qquad \frac{d}{dx}\left(\frac{y}{x}\right) = \cos x.$$

Integration gives

$$\frac{y}{x} = \sin x + C \qquad\Longrightarrow\qquad y(x) = x\,\sin x + Cx.$$

The initial condition requires $0 = 0+0$. Since this does not determine C, $y(x) = x\,\sin x + Cx$ is a solution of the initial-value problem for any value of C (except at $x=0$); that is, we have infinitely many solutions.

5. Find a general solution for the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = x + e^{-3x}.$$

The auxiliary equation $0 = m^2 + 6m + 9 = (m+3)^2$ has roots $m = -3, -3$. Hence,

$$y_h(x) = (C_1 + C_2 x)e^{-3x}.$$

When we substitute a particular solution $y_p(x) = Ax + B + Cx^2e^{-3x}$ into the differential equation,

$$C(2e^{-3x} - 12xe^{-3x} + 9x^2e^{-3x}) + 6(A + 2Cxe^{-3x} - 3Cx^2e^{-3x}) + 9(Ax + B + Cx^2e^{-3x}) = x + e^{-3x}.$$

Equating coefficients of x, 1, and e^{-3x} gives

$$9A = 1, \quad 6A + 9B = 0, \quad 2C = 1.$$

Thus, $y_p(x) = \dfrac{x}{9} - \dfrac{2}{27} + \dfrac{x^2}{2}e^{-3x}$, and

$$y(x) = (C_1 + C_2 x)e^{-3x} + \frac{x}{9} - \frac{2}{27} + \frac{x^2}{2}e^{-3x}.$$

5 **6.** A 20,000 litre tank contains 15 kilograms of salt dissolved in 10,000 litres of water. At time $t = 0$, a solution with 2 kilograms of salt per 100 litres of water starts entering the tank at 15 millilitres per minute. At the same time, well-stirred mixture is removed from the tank at 10 millilitres per minute. At the 3 minute mark, 5 kilograms of pure salt is added to the tank. Set up, but do **NOT** solve, an initial-value problem for the number of grams of salt in the tank at any given time. For what values of t is your differential equation valid?

Let $S(t)$ represent the number of grams of salt in the tank as a function of time t in minutes. Then

$$\frac{dS}{dt} = \frac{3}{10} + 5000\delta(t-3) - \frac{10S}{10^7 + 5t}, \qquad S(0) = 15,000.$$

The tank fills when

$$10^7 + 5t = 2 \times 10^7 \qquad \Longrightarrow \qquad t = \frac{10^7}{5}.$$

The differential equation is therefore valid for $0 < t < 10^7/5$ minutes.

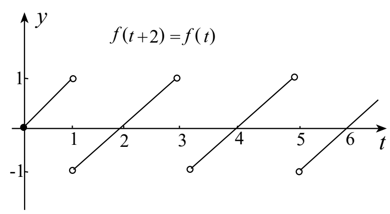
7. Find the Laplace transform for the function $f(t) = e^{-2t+3}\sin(4t+5)\,h(t-1)$.

$$\begin{aligned} F(s) &= e^{-s}\mathcal{L}\left\{e^{-2(t+1)+3}\sin[4(t+1)+5]\right\} = e^{-s}e\mathcal{L}\left\{e^{-2t}\sin(4t+9)\right\} \\ &= e^{1-s}\mathcal{L}\{\sin(4t+9)\}_{|s\to s+2} = e^{1-s}\mathcal{L}\{\sin 4t\cos 9 + \cos 4t \sin 9\}_{|s\to s+2} \\ &= e^{1-s}\left[\frac{4\cos 9}{(s+2)^2+16} + \frac{(\sin 9(s+2)}{(s+2)^2+16}\right] \end{aligned}$$

or,

$$
\begin{aligned}
F(s) &= e^3\mathcal{L}\left\{\sin(4t+5)h(t-1)\right\}_{|s\to s+2} = e^3\left[e^{-s}\mathcal{L}\left\{\sin\left[4(t+1)+5\right]\right\}\right]_{|s\to s+2}\\
&= e^3 e^{-(s+2)}\mathcal{L}\left\{\sin(4t+9)\right\}_{|s\to s+2} = e^{1-s}\mathcal{L}\left\{\sin 4t\cos 9 + \cos 4t \sin 9\right\}\\
&= e^{1-s}\left[\frac{4\cos 9}{(s+2)^2+16} + \frac{(\sin 9(s+2)}{(s+2)^2+16}\right]
\end{aligned}
$$

8. Find the Laplace transform for the function in the diagram below.



$$
\begin{aligned}
F(s) &= \frac{1}{1-e^{-2s}}\mathcal{L}\left\{t[h(t)-h(t-1)] + (t-2)[h(t-1)-h(t-2)]\right\}\\
&= \frac{1}{1-e^{-2s}}\mathcal{L}\left\{t - 2h(t-1) - (t-2)h(t-2)\right\}\\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - \frac{2e^{-s}}{s} - e^{-2s}\mathcal{L}\left\{(t+2)-2\right\}\right]\\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s^2}\right].
\end{aligned}
$$

8 **9** Find the inverse Laplace transform for the function

$$
F(s) = \frac{3s^2+7s+9}{s^3+3s^2+8s+6}.
$$

With the partial fraction decomposition

$$
\frac{3s^2+7s+9}{s^3+3s^2+8s+6} = \frac{1}{s+1} + \frac{2s+3}{s^2+2s+6},
$$

(which you had to derive),

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{2s+3}{s^2+2s+6}\right\} = e^{-t} + \mathcal{L}^{-1}\left\{\frac{2(s+1)+1}{(s+1)^2+5}\right\}\\
&= e^{-t} + e^{-t}\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+5}\right\} = e^{-t} + e^{-t}\left(2\cos\sqrt{5}t + \frac{1}{\sqrt{5}}\sin\sqrt{5}t\right).
\end{aligned}
$$

8 10. Find the inverse Laplace transform for the function

$$F(s) = \frac{e^{-s}}{s^3(1+e^{-s})}.$$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\sum_{n=0}^{\infty}(-e^{-s})^n\right\} = \sum_{n=0}^{\infty}\mathcal{L}^{-1}\left\{\frac{(-1)^n}{s^3}e^{-(n+1)s}\right\} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}(t-n-1)^2 h(t-n-1). \end{aligned}$$

12 11. A 1-kilogram mass is suspended from a spring with constant 10 newtons per metre. The mass is set into vertical motion by giving it velocity 1 metre per second downward. During its subsequent motion, it experiences damping equal in magnitude to 2 times its speed. After 1 second, it is struck upward by a hammer that imparts F units of momentum. Find the position of the mass as a function of time. Is there a sudden change in velocity of the mass as a result of the strike of the hammer? If so, how much is the change?

The initial-value problem for displacement $x(t)$ of the mass from equilibrium is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = F\delta(t-1), \qquad x(0) = 0, \quad x'(0) = -1.$$

When we take Laplace transforms,

$$(s^2X + 1) + 2(sX) + 10X = Fe^{-s}.$$

We solve for $X(s)$,

$$X(s) = \frac{-1 + Fe^{-s}}{s^2 + 2s + 10} = \frac{-1}{(s+1)^2 + 9} + \frac{Fe^{-s}}{(s+1)^2 + 9}.$$

Thus,

$$\begin{aligned} x(t) &= -e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + F\left[e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\right]_{|t\to t-1} h(t-1) \\ &= -\frac{1}{3}e^{-t}\sin 3t + \frac{F}{3}e^{-(t-1)}\sin 3(t-1)h(t-1) \text{ m}. \end{aligned}$$

The change in velocity is $F/1 = F$ m/s.

8 **12.** (a) When chemicals A and B are brought together, 1 gram of A reacts with 2 grams of B to make 3 grams of C. The rate at which C is formed is proportional to the product of the amounts of A and B present in the mixture. Initially, 10 grams of A and 30 grams of B are brought together. In addition, after 30 seconds, 5 grams of A are added to the mixture. Set up, but do **NOT** solve, an initial-value problem for the amount of C in the mixture at any given time.

(b) Can you apply techniques from Chapter 15 to solve this problem? Explain.

(c) Can you use Laplace transforms to solve the problem? Explain.

(a) Let $x(t)$ be the number of grams of C in the mixture at any given time t in seconds. Then,

$$\frac{dx}{dt} = k\left[10 - \frac{x}{3} + 5h(t-30)\right]\left[30 - \frac{2x}{3}\right], \qquad x(0) = 0.$$

(b) Yes. Chapter 15 can handle Heaviside functions, but with much difficulty.

(c) No. The differential equation is not linear.