Solutions to Fall 2023 Exam

1. Find the Taylor series about $x = 2$ for the function $f(x) = \dfrac{1}{\sqrt{3+2x}}$. Write your final answer in sigma notation, simplified as much as possible. You must use a method that guarantees that the series converges to $f(x)$. Determine the open interval of convergence of the series.

$\boxed{x-2}$

$$\begin{aligned}
\frac{1}{\sqrt{3+2x}} &= \frac{1}{\sqrt{7+2(x-2)}} = \frac{1}{\sqrt{7}}\left[1+\frac{2}{7}(x-2)\right]^{-1/2} \\
&= \frac{1}{\sqrt{7}}\left\{1+(-1/2)\frac{2}{7}(x-2)+\frac{(-1/2)(-3/2)}{2!}\left[\frac{2}{7}(x-2)\right]^2+\cdots\right\} \\
&= \frac{1}{\sqrt{7}}\left\{1-\frac{2}{2\cdot 7}(x-2)+\frac{(1\cdot 3)2^2}{2^2 7^2 2!}(x-2)^2+\cdots\right\} \\
&= \frac{1}{\sqrt{7}}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{7^n n!}(x-2)^n\right\} \\
&= \frac{1}{\sqrt{7}}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{2^n 7^n (n!)^2}(x-2)^n\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n 7^{n+1/2}(n!)^2}(x-2)^n
\end{aligned}$$

The open interval of convergence is

$$\left|\frac{2}{7}(x-2)\right| < 1 \quad\Longrightarrow\quad |x-2| < \frac{7}{2} \quad\Longrightarrow\quad -\frac{7}{2} < x-2 < \frac{7}{2} \quad\Longrightarrow\quad -\frac{3}{2} < x < \frac{11}{2}.$$

2. Determine whether the power series

$$\sum_{n=2}^{\infty} \frac{2^n}{(n+2)^2}(x+3)^n$$

converges at the left end of its open interval of convergence. Justify all statements.

The radius of convergence of the series is

$$R = \lim_{n\to\infty} \left| \frac{\frac{2^n}{(n+2)^2}}{\frac{2^{n+1}}{(n+3)^2}} \right| = \frac{1}{2}.$$

The open interval of convergence is therefore

$$|x+3| < \frac{1}{2} \implies -\frac{1}{2} < x+3 < \frac{1}{2} \implies -\frac{7}{2} < x < -\frac{5}{2}.$$

At the left end $x = -7/2$, the series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{(n+2)^2}\left(-\frac{1}{2}\right)^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n+2)^2}$. Since the sequence $\left\{\frac{1}{(n+2)^2}\right\}$ is decreasing and has limit 0, the series converges by the alternating series test.

3. Find a general solution for the differential equation

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = \cos 2x + 3x.$$

The auxiliary equation is

$$0 = m^3 - m^2 + m - 6 = (m-2)(m^2 + m + 3),$$

with solutions $m = 2$ and $m = \dfrac{-1 \pm \sqrt{1-12}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{11}i}{2}$. A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 e^{2x} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{11}x}{2} + C_3 \sin\frac{\sqrt{11}x}{2}\right).$$

When we substitute a particular solution $y_p(x) = A\cos 2x + B\sin 2x + Cx + D$ into the nonhomogeneous equation, we get

$$\begin{aligned}(8A\sin 2x - 8B\cos 2x) - (-4A\cos 2x - 4B\sin 2x) + (-2A\sin 2x + 2B\cos 2x + C)\\ - 6(A\cos 2x + B\sin 2x + Cx + D) = \cos 2x + 3x.\end{aligned}$$

We equate coefficients of linear independent terms,

$\cos 2x$: $-8B + 4A + 2B - 6A = 1$
$\sin 2x$: $8A + 4B - 2A - 6B = 0$
x : $-6C = 3$
1 : $C - 6D = 0.$

The solution is $A = -1/20$, $B = -3/20$, $C = -1/2$, and $D = -1/12$. A particular solution is

$$y_p(x) = -\frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x - \frac{x}{2} - \frac{1}{12}.$$

and a general solution of the given equation is

$$y(x) = C_1 e^{2x} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{11}x}{2} + C_3 \sin\frac{\sqrt{11}x}{2}\right) - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x - \frac{x}{2} - \frac{1}{12}.$$

4. A 3 kilogram mass hangs motionless on the end of a spring with constant 60 newtons per metre. It is set into vertical motion by raising it 10 centimetres and giving it velocity 20 centimetres per second downward. Assume that damping is negligible. If a vertical force $F(t) = 2\cos\omega t$ newtons is applied to the mass after 2 minutes, **set up, but do not solve** an initial-value problem for displacement of the mass from its equilibrium position. What positive value(s) of ω, if any, cause resonance?

The initial-value problem is

$$3\frac{d^2x}{dt^2} + 60x = 2\cos\omega t\, h(t-120), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -\frac{1}{5}.$$

The auxiliary equation $0 = 3m^2 + 60 = 3(m^2 + 20)$ has roots $m = \pm 2\sqrt{5}i$. A general solution of the associated homogeneous equation is $x_h(t) = C_1 \cos(2\sqrt{5}t) + C_2 \sin(2\sqrt{5}t)$. Resonance occurs when $\omega = 2\sqrt{5}$.

5. If the roots of the auxiliary equation for the linear differential equation

$$\phi(D)y = x^3 + e^{2x}\sin x + xe^x$$

are

$$m = 0, 0, 2 \pm i, 3 \pm 4i, 3 \pm \sqrt{7},$$

what is the form of a particular solution as predicted by undetermined coefficients?

A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 + C_2 x + e^{2x}(C_3\cos x + C_4 \sin x) + 3^{2x}(C_5\cos 4x + C_6 \sin 4x)$$
$$C_7 e^{(3+\sqrt{7})x} + C_8 e^{(3-\sqrt{7})x}.$$

A particular solution takess the form

$$y_p(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Exe^{2x}\sin x + Fxe^{2x}\cos x + Gxe^x + He^x.$$

6. Find Laplace transforms for the following functions:

(a) $f(t) = e^{-4t}\cos(2t+3)h(t-2)$ (b) $f(t) = \begin{cases} t, & 0 < t < 2, \\ t^2, & 2 < t < 5, \end{cases} \qquad f(t+5) = f(t)$

You need not simplify your answers.

(a)

$$\begin{aligned}
F(s) &= \mathcal{L}\{\cos(2t+3)\,h(t-2)\}_{|s\to s+4} = \left[e^{-2s}\mathcal{L}\{\cos[2(t+2)+3]\}\right]_{|s\to s+4} \\
&= e^{-2(s+4)}\mathcal{L}\{\cos 2t + 7)\}_{|s\to s+4} \\
&= e^{-2(s+4)}\mathcal{L}\{\cos 2t\cos 7 - \sin 2t \sin 7\}_{|s\to s+4} \\
&= e^{-2(s+4)}\left[\frac{s\,\cos 7}{s^2+4} - \frac{2\sin 7}{s^2+4}\right]_{|s\to s+4} \\
&= e^{-2(s+4}\left[\frac{(s+4)\cos 7}{(s+4^2+4} - \frac{2\sin 7}{(s+4)^2+4}\right]
\end{aligned}$$

or

$$\begin{aligned}
F(s) &= e^{-2s}\mathcal{L}\{e^{-4(t+2)}\cos[2(t+2)+3]\} \\
&= e^{-2s}\mathcal{L}\{e^{-8}e^{-4t}\cos(2t+7)\} \\
&= e^{-2(s+4)}\mathcal{L}\{\cos(2t+7)\}_{|s\to s+4} \\
&= e^{-2(s+4)}\mathcal{L}\{\cos 2t\cos 7 - \sin 2t\sin 7\}_{|s\to s+4} \\
&= e^{-2(s+4)}\left[\frac{s\,\cos 7}{s^2+4} - \frac{2\sin 7}{s^2+4}\right]_{|s\to s+4} \\
&= e^{-2(s+4}\left[\frac{(s+4)\cos 7}{(s+4^2+4} - \frac{2\sin 7}{(s+4)^2+4}\right]
\end{aligned}$$

(b)

$$\begin{aligned}
F(s) &= \frac{1}{1-e^{-5s}}\mathcal{L}\{t[h(t)-h(t-2)] + t^2[h(t-2)-h(t-5)]\} \\
&= \frac{1}{1-e^{-5s}}\mathcal{L}\{t + (t^2-t)h(t-2) - t^2 h(t-5)\} \\
&= \frac{1}{1-e^{-5s}}\left[\frac{1}{s^2} + e^{-2s}\mathcal{L}\{(t+2)^2-(t+2)\} - e^{-5s}\mathcal{L}\{(t+5)^2\}\right] \\
&= \frac{1}{1-e^{-5s}}\left[\frac{1}{s^2} + e^{-2s}\mathcal{L}\{t^2+3t+2\} - e^{-5s}\mathcal{L}\{t^2+10t+25\}\right] \\
&= \frac{1}{1-e^{-5s}}\left[\frac{1}{s^2} + e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right) - e^{-5s}\left(\frac{2}{s^3}+\frac{10}{s^2}+\frac{25}{s}\right)\right]
\end{aligned}$$

7. Find inverse Laplace transforms for the following functions:

(a) $F(s) = \dfrac{s+1}{s^3 - s^2 + 2s - 8}$ (b) $F(s) = \dfrac{s+1}{(s^2+2s)(1-e^{-4s})}$

(a)

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s^2+s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/10}{s-2} + \frac{-3s/10+1/10}{s^2+s+4}\right\} \\
&= \frac{3}{10}e^{2t} + \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{-3(s+1/2)+5/2}{(s+1/2)^2+15/4}\right\} \\
&= \frac{3}{10}e^{2t} + \frac{1}{10}e^{-t/2}\mathcal{L}^{-1}\left\{\frac{-3s+5/2}{s^2+15/4}\right\} \\
&= \frac{3}{10}e^{2t} + \frac{1}{10}e^{-t/2}\left(-3\cos\frac{\sqrt{15}t}{2} + \frac{5}{\sqrt{15}}\sin\frac{\sqrt{15}t}{2}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\left(\frac{1/2}{s} + \frac{1/2}{s+2}\right)\sum_{n=0}^{\infty}(e^{-4s})^n\right\} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} + \frac{1}{s+2}\right)e^{-4ns}\right\} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}[1+e^{-2(t-4n)}]h(t-4n).
\end{aligned}$$

8. Consider the following initial-value problem:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 2h(t-1) + 3\delta(t-2), \quad x(0) = 0, \quad x'(0) = 0.$$

(a) Describe a physical situation that would give rise to this problem.

(b) Find $x(t)$.

(a) The problem could arise from a mass of 1 kilogram suspended from a spring with constant 3 newtons per metre with damping 4 times velocity. It is orignally at rest at equilibrium. A force of 2 newtons acts upward for all time greater than or equal to 1 second, and it is struck with an impulse force of 3 newtons at time $t = 2$ seconds.

(b) When we take Laplace transforms,

$$s^2X + 4sX + 3X = \frac{2e^{-s}}{s} + 3e^{-2s},$$

from which

$$\begin{aligned} X(s) &= \frac{2e^{-s}/s + 3e^{-2s}}{s^2 + 4s + 3} = \frac{2e^{-s}}{s(s+3)(s+1)} + \frac{3e^{-2s}}{(s+3)(s+1)} \\ &= 2e^{-s}\left(\frac{1/3}{s} + \frac{1/6}{s+3} - \frac{1/2}{s+1}\right) + 3e^{-2s}\left(\frac{-1/2}{s+3} + \frac{1/2}{s+1}\right). \end{aligned}$$

Thus,

$$x(t) = 2\left[\frac{1}{3} + \frac{1}{6}e^{-3(t-1)} - \frac{1}{2}e^{-(t-1)}\right]h(t-1) + \frac{3}{2}\left[-e^{-3(t-2)} + e^{-(t-2)}\right]h(t-2).$$

9. Find an integral representation for the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = f(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

When we take Laplace transforms,

$$(s^2Y - 1) + 4sY + 6Y = F(s),$$

from which

$$\begin{aligned} Y(s) &= \frac{1 + F(s)}{s^2 + 4s + 6} = \frac{1}{s^2 + 4s + 6} + \frac{F(s)}{s^2 + 4s + 6} \\ &= \frac{1}{(s+2)^2 + 2} + \frac{F(s)}{(s+2)^2 + 2}. \end{aligned}$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 + 2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2}\right\} = \frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t,$$

$$y(t) = \frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t + \frac{1}{\sqrt{2}}\int_0^t e^{-2u}\sin\sqrt{2}u f(t-u)\,du.$$