Solutions to 2132 exam for summer 2021

5 **1.** Determine, with justification, whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{n+2}{n}\right)^n$$

Suppose we set $L=\lim_{x\to\infty}\left(\frac{x+2}{x}\right)^x$. If we take logarithms,

$$\ln L=\lim_{x\to\infty}\ln\left(\frac{x+2}{x}\right)^x=\lim_{x\to\infty}\left[x\ln\left(\frac{x+2}{x}\right)\right]=\lim_{x\to\infty}\frac{\ln\left(\frac{x+2}{x}\right)}{1/x}$$

$$=\lim_{x\to\infty}\frac{\frac{x}{x+2}\left(\frac{x(1)-(x+2)(1)}{x^2}\right]}{-1/x^2}=\lim_{x\to\infty}\left(\frac{2x}{x+2}\right)=2.$$

Hence $L=e^2$. It now follows that $\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n=e^2$, and the series diverges by the n^{th}-term test.

5 **2.** Determine values of x for which the following series converges and find its sum. Do **NOT** simplify the sum.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{3^{3n}}(x+2)^{2n}$$

Writing the series in the form $\sum_{n=2}^{\infty}\left[\frac{-(x+2)^2}{27}\right]^n$ shows that it is geometric, with sum

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{3^{3n}}(x+2)^{2n}=\frac{(x+2)^4/27^2}{1+\dfrac{(x+2)^2}{27}}.$$

The series converges for

$$\left|\frac{-(x+2)^2}{27}\right|<1 \implies (x+2)^2<27 \implies -\sqrt{27}<x+2<\sqrt{27} \implies -2-3\sqrt{3}<x<-2+3\sqrt{3}.$$

5 **3.** Find the Taylor series about $x = a$, where $a > 0$ is a constant, for the function $f(x) = \sin x$. Write your final answer in sigma notation, simplified as much as possible. You must use a method that guarantees that the series converges to $f(x)$.

x-a

$$\begin{aligned}\sin x &= \sin\left[(x-a)+a\right] = \sin(x-a)\cos a + \cos(x-a)\sin a \\ &= \cos a \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x-a)^{2n+1} + \sin a \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-a)^{2n}\end{aligned}$$

10 **4.** Find the Taylor series about $x = -3$ for the function

$$f(x) = \left(\frac{x+3}{x+5}\right)^2.$$

Write your final answer in sigma notation, simplified as much as possible. You must use a method that guarantees that the series converges to $f(x)$. What is the interval of convergence of the series? Hint: Do not divide $x+3$ by $x+5$.

x+3

We begin by writing the function in the form $f(x) = \dfrac{(x+3)^2}{(x+5)^2}$. Now, consider

$$\frac{1}{x+5} = \frac{1}{(x+3)+2} = \frac{1/2}{1+\frac{x+3}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x+3}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x+3)^n,$$

valid for

$$\left|\frac{-(x+3)}{2}\right| < 1 \quad\Longrightarrow\quad |x+3| < 2 \quad\Longrightarrow\quad -2 < x+3 < 2 \quad\Longrightarrow\quad -5 < x < -1.$$

With a positive radius of convergence, we can differentiate the series

$$\frac{-1}{(x+5)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{2^{n+1}}(x+3)^{n-1}.$$

Multiplication by $-(x+3)^2$ now yields

$$f(x) = \frac{(x+3)^2}{(x+5)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1} n}{2^{n+1}}(x+3)^{n+1} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{2^n}(x+3)^n.$$

Since differentiation never picks up an end point of the open interval interval of convergence, we can say that the interval of convergence is $-5 < x < -1$.

11 **5.** Solve the initial-value problem

$$y\frac{dy}{dx} + (1+y^2)\sin x = 0, \quad y(0) = 1.$$

There is a bonus of 2 marks if you find the domain of the solution?

The equation can be separated

$$\frac{y}{1+y^2}\,dy = -\sin x\,dx,$$

and therefore a one-parameter family of solutions is defined implicitly by

$$\frac{1}{2}\ln(1+y^2) = \cos x + C.$$

The initial condition requires $\frac{1}{2}\ln 2 = 1 + C \quad \Longrightarrow \quad C = \frac{1}{2}\ln 2 - 1.$

When we solve for $y(x)$, we obtain $y = \pm\sqrt{De^{2\cos x} - 1}$. Since the initial condition requires y to be positive at $x = 0$, we must choose the positive root, in which case the initial condition gives $1 = \sqrt{De^2 - 1}$, from which $D = 2e^{-2}$. Thus,

$$y(x) = \sqrt{2e^{2(\cos x - 1)} - 1}.$$

For $y(x)$ to be defined, we must have

$$2e^{2(\cos x-1)} - 1 \geq 0 \quad \Longrightarrow \quad \cos x \geq 1 - \frac{1}{2}\ln 2.$$

Thus, the solution is defined on the interval $|x| \leq \mathrm{Cos}^{-1}\left(1 - \frac{1}{2}\ln 2\right)$, but it is a solution of the differential equation only on the corresponding open interval.

5 **6.** Three chemicals A, B, and C combine to produce chemical D. Three grams of A, two grams of B, and one gram of C make 6 grams of D. The rate at which D is formed is proportional to the product of the amounts of A, B, and C in the mixture. If 100 grams of A, 20 grams of B, and 50 grams of C are brought together at time $t = 0$, set up, but do **NOT** solve, an initial-value problem for the number of grams of D in the mixture at any given time. Determine the amount of D in the mixture when reactions finish.

If $x(t)$ represents the number of grams of D in the mixture, then the initial-value problem for $x(t)$ is

$$\frac{dx}{dt} = k\left(100 - \frac{x}{2}\right)\left(20 - \frac{x}{3}\right)\left(50 - \frac{x}{6}\right), \quad x(0) = 0.$$

Reactions finish when $dx/dt = 0$, and this occurs when $x = 60$ grams.

9 **7.** The differential equation

$$2x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} - 4y = 2x^3$$

is linear, but it does not have constant coefficients.

(a) Show that a general solution of the associated homogeneous equation can be obtained by setting $y = x^m$ and finding values for m.

(b) Find a particular solution of the nonhomogeneous equation by setting $y_p(x) = Ax^3$, and finding A.

(c) What is a general solution of the nonhomogeneous equation?

(d) For what values of x is your solution valid?

(a) When we substitute $y = x^m$ into the homogeneous equation,

$$0 = 2x^2m(m-1)x^{m-2} - 5xmx^{m-1} - 4x^m = x^m(2m^2 - 7m - 4) = x^m(2m+1)(m-4).$$

Thus, $m = -1/2, 4$ and x^4 and $1/\sqrt{x}$ are solutions of the homogeneous equation. A general solution is $y_h(x) = C_1x^4 + C_2/\sqrt{x}$.

(b) When we substitute $y_p(x) = Ax^3$ into the nonhomogeneous equation

$$2x^3 = 2x^2(6Ax) - 5x(3Ax^2) - 4(Ax^3) = x^3(12A - 15A - 4A)x^3 = -7Ax^3 \quad \Longrightarrow \quad A = -\frac{2}{7}.$$

A particular solution is $y_p(x) = -2x^3/7$.

(c) A general solution of the nonhomogeneous equation is

$$y(x) = C_1x^4 + C_2/\sqrt{x} - \frac{2x^3}{7}.$$

(d) The solution is valid for $x > 0$.

10 **8.** Find Laplace transforms for the following functions:

(a) $f(t) = \cos(4t+1)h(t-1)$ (b) $f(t) = 4 - t^2, \quad 0 < t < 2, \quad f(t+2) = f(t)$

(a)

$$\begin{aligned} F(s) &= e^{-s}\mathcal{L}\{\cos[4(t+1)+1]\} = e^{-s}\mathcal{L}\{\cos(4t+5)\} \\ &= e^{-s}\mathcal{L}\{\cos 4t \cos 5 - \sin 4t \sin 5\} = e^{-s}\left[\frac{(\cos 5)s}{s^2+16} - \frac{4\sin 5}{s^2+16}\right] \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-2s}}\mathcal{L}\{(4-t^2)[h(t)-h(t-2)]\} \\ &= \frac{1}{1-e^{-2s}}[\mathcal{L}\{4-t^2\} - e^{-2s}\mathcal{L}\{4-(t+2)^2\} \\ &= \frac{1}{1-e^{-2s}}\left[\frac{4}{s} - \frac{2}{s^3} - e^{-2s}\mathcal{L}\{-t^2-4t\}\right] \\ &= \frac{1}{1-e^{-2s}}\left[\frac{4}{s} - \frac{2}{s^3} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right)\right] \end{aligned}$$

14 **9.** Find inverse Laplace transforms for the following functions:

(a) $F(s) = \dfrac{1}{(s^2+s)(1+e^{-2s})}$ (b) $F(s) = \dfrac{s+2}{2s^2+5s+4}$

(a)

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{1}{s+1}\right)\sum_{n=0}^{\infty}(-e^{-2s})^n\right\} = \sum_{n=0}^{\infty}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{1}{s+1}\right)(-1)^n e^{-2ns}\right\} \\ &= \sum_{n=0}^{\infty}(-1)^n[1-e^{-(t-2n)}]h(t-2n) \end{aligned}$$

(b)

$$\begin{aligned} f(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+5s/2+2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(s+5/4)+2-5/4}{(s+5/4)^2+2-25/16}\right\} \\ &= \frac{1}{2}e^{-5t/4}\mathcal{L}^{-1}\left\{\frac{s+3/4}{s^2+7/16}\right\} = \frac{1}{2}e^{-5t/4}\left(\cos\frac{\sqrt{7}t}{4} + \frac{3}{\sqrt{7}}\sin\frac{\sqrt{7}t}{4}\right) \end{aligned}$$

12 10. Find an integral representation for a general solution of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = f(t),$$

where $f(t)$ is an unspecified function.

If we set $y(0) = A$ and $y'(0) = B$, and take Laplace transforms

$$(s^2Y - As - B) + 4(sY - A) + 4Y = F(s).$$

When we solve for $Y(s)$,

$$Y(s) = \frac{As + (4A + B)}{s^2 + 4s + 4} + \frac{F(s)}{s^2 + 4s + 4} = \frac{As + (4A + B)}{(s + 2)^2} + \frac{F(s)}{(s + 2)^2}.$$

The partial fraction decomposition of the first term is of the form $\dfrac{C}{s + 2} + \dfrac{D}{(s + 2)^2}$.

Since $\mathcal{L}^{-1}\left\{\dfrac{1}{(s + 2)^2}\right\} = te^{-2t}$, we can use convolutions to write

$$y(t) = Ce^{-2t} + De^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \int_0^t ue^{-2u}f(t - u)\,du = Ce^{-2t} + Dte^{-2t} + \int_0^t ue^{-2u}f(t - u)\,du.$$

14 11. A 1-kilogram mass is suspended from a spring with constant 100 newtons per metre. The mass is set into vertical motion by lifting it 10 centimetres above its equilibrium position and releasing it. During its subsequent motion, it experiences damping equal in magnitude to 4 times its speed. If, in addition, it is struck by a hammer at time $t = 2$ seconds that imparts 3 units of momentum upward, find its displacement for all time. Is there a sudden change in velocity of the mass as a result of the strike of the hammer? If so, how much is the change?

The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 100x = 3\delta(t - 2), \quad x(0) = \frac{1}{10}, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$\left(s^2X - \frac{s}{10}\right) + 4\left(sX - \frac{1}{10}\right) + 100X = 3e^{-2s} \quad \Longrightarrow \quad X(s) = \frac{s/10 + 2/5}{s^2 + 4s + 100} + \frac{3e^{-2s}}{s^2 + 4s + 100}.$$

The inverse transform is

$$\begin{aligned} x(t) &= \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{(s + 2) + 2}{(s + 2)^2 + 96}\right\} + 3\mathcal{L}^{-1}\left\{\frac{e^4e^{-2(s+2)}}{(s + 2)^2 + 96}\right\} \\ &= \frac{1}{10}e^{-2t}\mathcal{L}^{-1}\left\{\frac{s + 2}{s^2 + 96}\right\} + 3e^4e^{-2t}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2 + 96}\right\} \\ &= \frac{1}{10}e^{-2t}\left(\cos 4\sqrt{6}t + \frac{\sqrt{6}}{12}\sin 4\sqrt{6}t\right) + \frac{\sqrt{6}}{8}e^{-2(t-2)}\sin 4\sqrt{6}(t - 2)h(t - 2). \end{aligned}$$

There is an abrupt change in velocity of $3/1 = 3$ m/s.