CHAPTER 6

EXERCISES 6.1

1. $\mathcal{L}\{t^3-2t^2+1\}=\mathcal{L}\{t^3\}-2\mathcal{L}\{t^2\}+\mathcal{L}\{1\}=\dfrac{3!}{s^4}-2\left(\dfrac{2!}{s^3}\right)+\dfrac{1}{s}=\dfrac{6}{s^4}-\dfrac{4}{s^3}+\dfrac{1}{s}$

2. $\mathcal{L}\{t+e^t\}=\mathcal{L}\{t\}+\mathcal{L}\{e^t\}=\dfrac{1}{s^2}+\dfrac{1}{s-1}$

3. $\mathcal{L}\{5e^{4t}\}=5\mathcal{L}\{e^{4t}\}=\dfrac{5}{s-4}$

4. $\mathcal{L}\{e^{-2t}+2e^t\}=\mathcal{L}\{e^{-2t}\}+2\mathcal{L}\{e^t\}=\dfrac{1}{s+2}+\dfrac{2}{s-1}$

5. $\mathcal{L}\{\sin 4t+3\cos 4t\}=\mathcal{L}\{\sin 4t\}+3\mathcal{L}\{\cos 4t\}=\dfrac{4}{s^2+16}+\dfrac{3s}{s^2+16}=\dfrac{3s+4}{s^2+16}$

6. $\{\cos 2t-3\sin 4t\}=\mathcal{L}\{\cos 2t\}-3\mathcal{L}\{\sin 4t\}=\dfrac{s}{s^2+4}-\dfrac{12}{s^2+16}$

7. $\mathcal{L}\{5t\cos 2t\}=5\mathcal{L}\{t\cos 2t\}=\dfrac{5(s^2-4)}{(s^2+4)^2}$

8. $\mathcal{L}\{3t\sin 4t\}=3\mathcal{L}\{t\sin 4t\}=3\left[\dfrac{2(4)s}{(s^2+16)^2}\right]=\dfrac{24s}{(s^2+16)^2}$

9. $\mathcal{L}\{5t\cos t-2t\sin t\}=5\mathcal{L}\{t\cos t\}-2\mathcal{L}\{t\sin t\}=5\left[\dfrac{s^2-1}{(s^2+1)^2}\right]-2\left[\dfrac{2s}{(s^2+1)^2}\right]=\dfrac{5s^2-4s-5}{(s^2+1)^2}$

10. $\mathcal{L}\{3t\sin t-\cos t\}=3\mathcal{L}\{t\sin t\}-\mathcal{L}\{\cos t\}=3\left[\dfrac{2s}{(s^2+1)^2}\right]-\dfrac{s}{s^2+1}=\dfrac{5s-s^3}{(s^2+1)^2}$

11. $\mathcal{L}^{-1}\left\{\dfrac{7}{s^3}\right\}=7\mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\right\}=7\left(\dfrac{t^2}{2}\right)=\dfrac{7t^2}{2}$

12. $\mathcal{L}^{-1}\left\{\dfrac{2}{s}-\dfrac{3}{s^4}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\}-3\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\}=2(1)-3\left(\dfrac{t^3}{3!}\right)=2-\dfrac{t^3}{2}$

13. $\mathcal{L}^{-1}\left\{\dfrac{1}{s+5}+\dfrac{4}{s^2}\right\}=\mathcal{L}^{-1}\left\{\dfrac{1}{s+5}\right\}+4\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}=e^{-5t}+4t$

14. $\mathcal{L}^{-1}\left\{\dfrac{3}{s-1}\right\}=3\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}=3e^t$

15. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}-\dfrac{3}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}-3\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\}=\cos 2t-3\left(\dfrac{1}{2}\sin 2t\right)$

$=\cos 2t-\dfrac{3}{2}\sin 2t$

16. $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+2}-\dfrac{5}{s^2+9}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2}\right\}-5\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\}=2\cos\sqrt{2}t-5\left(\dfrac{1}{3}\sin 3t\right)$

$=2\cos\sqrt{2}t-\dfrac{5}{3}\sin 3t$

17. $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2+2)^2}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+2)^2}\right\}=2\left(\dfrac{t}{2\sqrt{2}}\sin\sqrt{2}t\right)=\dfrac{t}{\sqrt{2}}\sin\sqrt{2}t$

18. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+9)^2}\right\}=\dfrac{1}{6}(\sin 3t+3t\cos 3t)=\dfrac{1}{6}\sin 3t+\dfrac{t}{2}\cos 3t$

19. $\mathcal{L}^{-1}\left\{\dfrac{3s-s^2}{(s^2+4)^2}\right\}=3\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\}-\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}=3\left(\dfrac{t}{4}\sin 2t\right)-\dfrac{1}{4}(\sin 2t+2t\cos 2t)$

$=\dfrac{1}{4}(3t-1)\sin 2t-\dfrac{t}{2}\cos 2t$

20. $$\mathcal{L}^{-1}\left\{\frac{s^2-2}{(s^2+3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+3)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+3)^2}\right\}$$
$$= \frac{1}{2\sqrt{3}}(\sin\sqrt{3}t + \sqrt{3}t\cos\sqrt{3}t) - \frac{2}{6\sqrt{3}}(\sin\sqrt{3}t - \sqrt{3}t\cos\sqrt{3}t)$$
$$= \frac{1}{6\sqrt{3}}\sin\sqrt{3}t + \frac{5t}{6}\cos\sqrt{3}t$$

21. $F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_3^\infty e^{-st}\,dt = \left\{\frac{e^{-st}}{-s}\right\}_3^\infty = \frac{e^{-3s}}{s}$, provided $s>0$

22. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^4 e^{-st}\,dt + \int_4^\infty 2e^{-st}\,dt = \left\{\frac{e^{-st}}{-s}\right\}_0^4 + 2\left\{\frac{e^{-st}}{-s}\right\}_4^\infty$$
$$= \left(-\frac{e^{-4s}}{s} + \frac{1}{s}\right) + \frac{2e^{-4s}}{s} = \frac{1+e^{-4s}}{s}, \quad \text{provided } s>0$$

23. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^2 te^{-st}\,dt + \int_2^\infty 2e^{-st}\,dt = \left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_0^2 + 2\left\{\frac{e^{-st}}{-s}\right\}_2^\infty$$
$$= \left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}\right) + 2\left(\frac{e^{-2s}}{s}\right) = \frac{1}{s^2}(1-e^{-2s}), \quad \text{provided } s>0$$

24. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 t^2e^{-st}\,dt = \left\{-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_0^1$$
$$= \frac{2}{s^3} - \frac{e^{-s}}{s^3}(s^2+2s+2), \quad \text{provided } s>0$$

25. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_1^\infty t^2e^{-st}\,dt = \left\{-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_1^\infty$$
$$= \frac{e^{-s}}{s^3}(s^2+2s+2), \quad \text{provided } s>0$$

26. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_1^\infty (t-1)^2e^{-st}\,dt = \left\{-\frac{(t-1)^2}{s}e^{-st} - \frac{2(t-1)}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_1^\infty$$
$$= \frac{2}{s^3}e^{-s}, \quad \text{provided } s>0$$

27. $F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_1^2 e^{-st}\,dt = \left\{\frac{e^{-st}}{-s}\right\}_1^2 = \frac{e^{-s}-e^{-2s}}{s}$, provided $s>0$

28. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 te^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt$$
$$= \left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_0^1 + \left\{\frac{t-2}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right\}_1^2 = \frac{1-2e^{-s}+e^{-2s}}{s^2}, \quad \text{provided } s>0$$

29. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 2te^{-st}\,dt + \int_1^\infty te^{-st}\,dt$$
$$= 2\left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_0^1 + \left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_1^\infty = \frac{2}{s^2} - \frac{(s+1)e^{-s}}{s^2}, \quad \text{provided } s>0$$

30. $$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 (1+t^2)e^{-st}\,dt + \int_1^\infty 2te^{-st}\,dt$$
$$= \left\{\frac{e^{-st}}{-s} - \frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_0^1 + 2\left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_1^\infty$$
$$= \frac{1}{s} + \frac{2(1-e^{-s})}{s^3}, \quad \text{provided } s>0$$

31. $F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_a^\infty e^{-st}\, dt = \left\{\frac{e^{-st}}{-s}\right\}_a^\infty = \frac{1}{s}e^{-as}$, provided $s > 0$

32. $F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_a^b e^{-st}\, dt = \left\{\frac{e^{-st}}{-s}\right\}_a^b = \frac{e^{-as} - e^{-bs}}{s}$, provided $s > 0$.

33. According to equation 6.2, the Laplace transform of $\sin at$ is

$$F(s) = \int_0^\infty e^{-st} \sin at\, dt.$$

Integration by parts with $u = \sin at$, $du = a\cos at\, dt$, $dv = e^{-st}\, dt$, and $v = -(1/s)e^{-st}$, gives

$$F(s) = \left\{-\frac{1}{s}e^{-st}\sin at\right\}_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st} a\cos at\, dt = \frac{a}{s}\int_0^\infty e^{-st}\cos at\, dt,$$

provided $s > 0$. A second integration by parts with $u = \cos at$, $du = -a\sin at\, dt$, $dv = e^{-st}\, dt$, and $v = -(1/s)e^{-st}$, yields

$$F(s) = \frac{a}{s}\left\{-\frac{1}{s}e^{-st}\cos at\right\}_0^\infty - \frac{a}{s}\int_0^\infty -\frac{1}{s}e^{-st}(-a\sin at)\, dt = \frac{a}{s^2} - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin at\, dt,$$

provided once again that $s > 0$. We can therefore write that

$$F(s) = \frac{a}{s^2} - \frac{a^2}{s^2}F(s),$$

and when this equation is solved for $F(s)$, the result is $F(s) = a/(s^2 + a^2)$. A similar derivation gives the transform of $\cos at$.

34. An unpalatable thing to do would be to use the definition of the transform as an improper integral. Better would be to consider the integral

$$\int_0^\infty te^{ati}e^{-st}dt = \int_0^\infty te^{(-s+ai)t}dt.$$

Integration by parts with $u = t$, $du = dt$, $dv = e^{(-s+ai)t}dt$, and $v = e^{(-s+ai)t}/(-s + ai)$, gives

$$\int_0^\infty te^{ati}e^{-st}dt = \left\{\frac{te^{(-s+ai)t}}{-s + ai}\right\}_0^\infty - \int_0^\infty \frac{e^{(-s+ai)t}}{-s + ai}dt.$$

Now,

$$\lim_{t\to\infty}\left[\frac{te^{(-s+ai)t}}{-s + ai}\right] = \lim_{t\to\infty}\left[\frac{te^{-st}(\cos at + i\sin at)}{-s + ai}\right] = 0,$$

provided $s > 0$. Hence,

$$\int_0^\infty te^{ati}e^{-st}dt = -\left\{\frac{e^{(-s+ai)t}}{(-s + ai)^2}\right\}_0^\infty = \frac{1}{(-s + ai)^2} = \frac{(s + ai)^2}{(s - ai)^2(s + ai)^2} = \frac{s^2 - a^2 + 2asi}{(s^2 + a^2)^2}.$$

When we take real and imaginary parts,

$$\int_0^\infty t\cos at\, e^{-st}dt = \frac{s^2 - a^2}{(s^2 + a^2)^2}, \qquad \int_0^\infty t\sin at\, e^{-st}dt = \frac{2as}{s^2 + a^2};$$

that is,

$$\mathcal{L}\{t\cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}, \qquad \mathcal{L}\{t\sin at\} = \frac{2as}{(a^2 + a^2)^2}.$$

35. If we set $u = \sqrt{t}$, or, $t = u^2$ in $F(s) = \int_0^\infty \frac{1}{\sqrt{t}}e^{-st}\, dt$, we find

$$F(s) = \int_0^\infty \frac{1}{u} e^{-su^2}(2u\,du) = 2\int_0^\infty e^{-su^2}\,du.$$

We now set $v = \sqrt{s}u$, in which case

$$F(s) = 2\int_0^\infty e^{-v^2}\left(\frac{dv}{\sqrt{s}}\right) = \frac{2}{\sqrt{s}}\int_0^\infty e^{-v^2}\,dv = \frac{2}{\sqrt{s}}\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\frac{\pi}{s}}.$$

36. Yes, they are of exponential order $O(e^{0t})$.

37. No. The function e^{t^2} is continuous, but not of exponential order.

38. When we take the Laplace transform of both sides of the equation and take the transform of the series term-by-term, we get

$$\mathcal{L}\{\sin at\} = \sum_{n=0}^\infty \frac{(-1)^n a^{2n+1}}{(2n+1)!}\mathcal{L}\{t^{2n+1}\} = \sum_{n=0}^\infty \frac{(-1)^n a^{2n+1}}{(2n+1)!}\left[\frac{(2n+1)!}{s^{2n+2}}\right] = \sum_{n=0}^\infty \frac{a}{s^2}\left(-\frac{a^2}{s^2}\right)^n.$$

This is a geometric series with common ratio $-a^2/s^2$, and therefore the sum is

$$\mathcal{L}\{\sin at\} = \frac{a/s^2}{1+a^2/s^2} = \frac{a}{s^2+a^2}.$$

39. The Maclaurin series for $\cos at$ is

$$\cos at = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}(at)^{2n}.$$

When we take the Laplace transform of both sides of the equation and take the transform of the series term-by-term, we get

$$\mathcal{L}\{\cos at\} = \sum_{n=0}^\infty \frac{(-1)^n a^{2n}}{(2n)!}\mathcal{L}\{t^{2n}\} = \sum_{n=0}^\infty \frac{(-1)^n a^{2n}}{(2n)!}\left[\frac{(2n)!}{s^{2n+1}}\right] = \sum_{n=0}^\infty \frac{1}{s}\left(-\frac{a^2}{s^2}\right)^n.$$

This is a geometric series with common ratio $-a^2/s^2$, and therefore the sum is

$$\mathcal{L}\{\sin at\} = \frac{1/s}{1+a^2/s^2} = \frac{s}{s^2+a^2}.$$

40. When we take the Laplace transform of both sides of the equation and take the transform of the series term-by-term, we get

$$\mathcal{L}\{J_0(t)\} = \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n}(n!)^2}\mathcal{L}\{t^{2n}\} = \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n}(n!)^2}\left[\frac{(2n)!}{s^{2n+1}}\right] = \sum_{n=0}^\infty \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\left(\frac{1}{s^{2n+1}}\right).$$

We can use the binomial expansion to write

$$\begin{aligned}
\frac{1}{\sqrt{1+s^2}} &= \frac{1}{s\sqrt{1+1/s^2}} = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2} \\
&= \frac{1}{s}\left[1 + \frac{-1/2}{s^2} + \frac{(-1/2)(-3/2)}{2!s^4} + \frac{(-1/2)(-3/2)(-5/2)}{3!s^6} + \cdots\right] \\
&= \frac{1}{s}\left[1 - \frac{1}{2s^2} + \frac{1\cdot 3}{2^2\,2!s^4} - \frac{1\cdot 3\cdot 5}{2^3\,3!s^6} + \cdots\right] \\
&= \frac{1}{s}\left[1 + \sum_{n=1}^\infty \frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n\,n!s^{2n}}\right] \\
&= \frac{1}{s}\left[1 + \sum_{n=1}^\infty \frac{(-1)^n[1\cdot 2\cdot 3\cdots(2n-1)(2n)]}{2^n\,n![2\cdot 4\cdot 6\cdots(2n)]s^{2n}}\right] = \sum_{n=0}^\infty \frac{(-1)^n(2n)!}{2^{2n}(n!)^2 s^{2n+1}}.
\end{aligned}$$

Hence, $\mathcal{L}\{J_0(t)\} = \dfrac{1}{\sqrt{1+s^2}}$.

41. When we expand $\sin at$ in its Maclaurin series, and take Laplace transforms term-by-term

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \mathcal{L}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)!}t^{2n}\right\} = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)!}\left[\frac{(2n)!}{s^{2n+1}}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)s^{2n+1}}.$$

If we expand $1/(a^2+s^2)$ in a geometric series, we obtain

$$\frac{1}{a^2+s^2} = \frac{1/s^2}{1+a^2/s^2} = \frac{1}{s^2}\sum_{n=0}^{\infty}\left(-\frac{a^2}{s^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{s^{2n+2}}.$$

Term-by-term integration with respect to a gives

$$\frac{1}{s}\mathrm{Tan}^{-1}\left(\frac{a}{s}\right) + C = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)s^{2n+2}}.$$

Substitution of $a=0$ gives $C=0$, and therefore

$$\mathrm{Tan}^{-1}\left(\frac{a}{s}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)s^{2n+1}}.$$

Thus, $\mathcal{L}\left\{\dfrac{\sin at}{t}\right\} = \mathrm{Tan}^{-1}\left(\dfrac{a}{s}\right)$.

42. When the exponentials are expanded in Maclaurin series,

$$\mathcal{L}\left\{\frac{e^{bt}-e^{at}}{t}\right\} = \mathcal{L}\left\{\sum_{n=1}^{\infty}\frac{b^n-a^n}{n!}t^{n-1}\right\} = \sum_{n=1}^{\infty}\frac{b^n-a^n}{n!}\left[\frac{(n-1)!}{s^n}\right] = \sum_{n=1}^{\infty}\frac{b^n-a^n}{ns^n}.$$

Expansion of $1/(s-a)$ in a geometric series leads to

$$\frac{1}{s-a} = \frac{1/s}{1-a/s} = \frac{1}{s}\sum_{n=0}^{\infty}\frac{a^n}{s^n} = \sum_{n=0}^{\infty}\frac{a^n}{s^{n+1}}.$$

Term-by-term integration with respect to a gives

$$-\ln|s-a| + C = \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)s^{n+1}}.$$

Evaluation at $a=0$ implies that $C=\ln|s|$. It now follows that for $s>b>a$,

$$\ln\left(\frac{s-a}{s-b}\right) = \ln(s-a) - \ln(s-b) = \left[\ln|s| - \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)s^{n+1}}\right] - \left[\ln|s| - \sum_{n=0}^{\infty}\frac{b^{n+1}}{(n+1)s^{n+1}}\right]$$
$$= \sum_{n=0}^{\infty}\frac{b^{n+1}-a^{n+1}}{(n+1)s^{n+1}} = \sum_{n=1}^{\infty}\frac{b^n-a^n}{ns^n}.$$

Hence, $\mathcal{L}\left\{\dfrac{e^{bt}-e^{at}}{t}\right\} = \ln\left(\dfrac{s-a}{s-b}\right)$.

43. (a) To prove this recursive formula, we use integration by parts, and to facilitate this we change the variable of integration,

$$\Gamma(t+1) = \int_0^{\infty} e^{-x}x^t\,dx.$$

If we set $u=x^t$, $dv=e^{-x}dx$, $du=tx^{t-1}$, nd $v=-e^{-x}$,

$$\Gamma(t+1) = \left\{-x^t e^{-x}\right\}_0^\infty - \int_0^\infty -tx^{t-1}e^{-x}\,dx = t\int_0^\infty e^{-x}x^{t-1}\,dx = t\,\Gamma(t).$$

When t is an integer n, and we iterate the recursive formula,

$$\Gamma(n+1) = n\,\Gamma(n) = \cdots = n(n-1)(n-2)\cdots(1)\Gamma(1) = n!\int_0^\infty e^{-x}\,dx = n!\left\{-e^{-x}\right\}_0^\infty = n!.$$

(b) If we set $u = st$ in the definition of $\mathcal{L}\{t^r\}$,

$$\mathcal{L}\{t^r\} = \int_0^\infty e^{-st}t^r\,dt = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^r\left(\frac{du}{s}\right) = \frac{1}{s^{r+1}}\int_0^\infty e^{-u}u^r\,du = \frac{\Gamma(r+1)}{s^{r+1}}.$$

44. If e^{t^2} is $O(e^{\alpha t})$ for some α, there must exist constants M and T such that for all $t > T$, the function e^{t^2} would have to be less than $Me^{\alpha t} = e^{\alpha t + \ln M}$. This would mean that t^2 would have to be less that $\alpha t + \ln M$ for all $t > T$. But this is impossible. No matter what the value of α, t^2 will eventually be larger than $\alpha t + \ln M$.

45. (a) Since $f(t)$ is bounded, it is of exponential order (Exercise 36).

(b) $f'(t) = 2te^{t^2}\cos(e^{t^2})$ For any value T, there are values of $t > T$ for which $\cos(e^{t^2}) = 1$, and for these values of t, the values of $f'(t)$ are equal to $2te^{t^2}$. Since e^{t^2} is not of exponential order, it follows that $f'(t)$ cannot be of exponential order.

EXERCISES 6.2

1. Since $f(t) = h(t-3)$, $F(s) = \mathcal{L}\{h(t-3)\} = \dfrac{e^{-3s}}{s}$.

2. Since $f(t) = [h(t) - h(t-4)] + 2h(t-4) = 1 + h(t-4)$,

$$F(s) = \mathcal{L}\{1 + h(t-4)\} = \frac{1}{s} + \frac{e^{-4s}}{s} = \frac{1 + e^{-4s}}{s}.$$

3. Since $f(t) = t[h(t) - h(t-2)] + 2h(t-2) = t + (2-t)h(t-2)$,

$$F(s) = \mathcal{L}\{t + (2-t)h(t-2)\} = \frac{1}{s^2} + e^{-2s}\mathcal{L}\{2 - (t+2)\} = \frac{1}{s^2} + e^{-2s}\mathcal{L}\{-t\} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} = \frac{1 - e^{-2s}}{s^2}.$$

4. Since $f(t) = t^2[h(t) - h(t-1)] = t^2 - t^2h(t-1)$,

$$F(s) = \mathcal{L}\{t^2 - t^2h(t-1)\} = \frac{2}{s^3} - e^{-s}\mathcal{L}\{(t+1)^2\} = \frac{2}{s^3} - e^{-s}\mathcal{L}\{t^2 + 2t + 1\}$$
$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) = \frac{2}{s^3} - \frac{e^{-s}(s^2 + 2s + 2)}{s^3}.$$

5. Since $f(t) = t^2h(t-1)$,

$$F(s) = \{t^2h(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}\mathcal{L}\{t^2 + 2t + 1\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) = \frac{e^{-s}(s^2 + 2s + 2)}{s^3}.$$

6. Since $f(t) = (t-1)^2h(t-1)$,

$$F(s) = \mathcal{L}\{(t-1)^2h(t-1)\} = e^{-s}\mathcal{L}\{(t+1-1)^2\} = e^{-s}\mathcal{L}\{t^2\} = \frac{2e^{-s}}{s^3}.$$

7. Since $f(t) = h(t-1) - h(t-2)$,

$$F(s) = \mathcal{L}\{h(t-1) - h(t-2)\} = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} = \frac{e^{-s} - e^{-2s}}{s}.$$

8. Since $f(t) = t[h(t) - h(t-1)] + (2-t)[h(t-1) - h(t-2)] = t + (2-2t)h(t-1) + (t-2)h(t-2)$,

$$F(s) = \mathcal{L}\{t + (2-2t)h(t-1) + (t-2)h(t-2)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{2 - 2(t+1)\} + e^{-2s}\mathcal{L}\{(t+2) - 2\}$$
$$= \frac{1}{s^2} + e^{-s}\mathcal{L}\{-2t\} + e^{-2s}\mathcal{L}\{t\} = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}.$$

9. Since $f(t) = 2t[h(t) - h(t-1)] + t\,h(t-1) = 2t - t\,h(t-1)$,

$$F(s) = \mathcal{L}\{2t - t\,h(t-1)\} = \frac{2}{s^2} - e^{-s}\mathcal{L}\{t+1\} = \frac{2}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = \frac{2}{s^2} - \frac{(s+1)e^{-s}}{s^2}.$$

10. Since $f(t) = (1+t^2)[h(t) - h(t-1)] + 2t\,h(t-1) = 1 + t^2 + (2t - 1 - t^2)h(t-1)$,

$$F(s) = \mathcal{L}\{1 + t^2 + (2t - 1 - t^2)h(t-1)\} = \frac{1}{s} + \frac{2}{s^3} + e^{-s}\mathcal{L}\{2(t+1) - 1 - (t+1)^2\}$$
$$= \frac{1}{s} + \frac{2}{s^3} + e^{-s}\mathcal{L}\{-t^2\} = \frac{1}{s} + \frac{2}{s^3} - \frac{2}{s^3}e^{-s} = \frac{1}{s} + \frac{2(1 - e^{-s})}{s^3}.$$

11. Since $f(t) = h(t-a)$, $F(s) = \mathcal{L}\{h(t-a)\} = \dfrac{e^{-as}}{s}$.

12. Since $f(t) = h(t-a) - h(t-b)$, $F(s) = \mathcal{L}\{h(t-a) - h(t-b)\} = \dfrac{e^{-as}}{s} - \dfrac{e^{-bs}}{s} = \dfrac{e^{-as} - e^{-bs}}{s}$.

13. Since $f(t) = 2(1-t)[h(t) - h(t-1)] = 2 - 2t + (2t-2)h(t-1)$,

$$F(s) = \mathcal{L}\{2 - 2t + (2t-2)h(t-1)\} = \frac{2}{s} - \frac{2}{s^2} + e^{-s}\mathcal{L}\{2(t+1) - 2\} = \frac{2}{s} - \frac{2}{s^2} + e^{-s}\mathcal{L}\{2t\}$$
$$= \frac{2}{s} - \frac{2}{s^2} + \frac{2e^{-s}}{s^2} = \frac{2}{s} + \frac{2(e^{-s}-1)}{s^2}.$$

14. Since $f(t) = 2[h(t) - h(t-1)] + [h(t-1) - h(t-2)] + (t-2)h(t-2) = 2 - h(t-1) + (t-3)h(t-2)$,

$$F(s) = \mathcal{L}\{2 - h(t-1) + (t-3)h(t-2)\} = \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\mathcal{L}\{(t+2) - 3\} = \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\mathcal{L}\{t-1\}$$
$$= \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right) = \frac{2 - e^{-s}}{s} + \frac{(1-s)e^{-2s}}{s^2}.$$

15. Since $f(t) = 4(1-t^2)[h(t) - h(t-1)] = 4 - 4t^2 + 4(t^2-1)h(t-1)$,

$$F(s) = \mathcal{L}\{4 - 4t^2 + 4(t^2-1)h(t-1)\} = \frac{4}{s} - \frac{8}{s^3} + 4e^{-s}\mathcal{L}\{(t+1)^2 - 1\} = \frac{4}{s} - \frac{8}{s^3} + 4e^{-s}\mathcal{L}\{t^2 + 2t\}$$
$$= \frac{4}{s} - \frac{8}{s^3} + 4e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) = \frac{4(s^2-2)}{s^3} + \frac{8(s+1)e^{-s}}{s^3}.$$

16. Since $f(t) = (1-t)[h(t) - h(t-1)] + (t-1)^2[h(t-1) - h(t-2)] = 1 - t + (t^2-t)h(t-1) - (t-1)^2h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{1 - t + (t^2-t)h(t-1) - (t-1)^2h(t-2)\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{(t+1)^2 - (t+1)\} - e^{-2s}\mathcal{L}\{(t+2-1)^2\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{t^2 + t\} - e^{-2s}\mathcal{L}\{t^2 + 2t + 1\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \\ &= \frac{s-1}{s^2} + \frac{(s+2)e^{-s}}{s^3} - \frac{(s^2+2s+2)e^{-2s}}{s^3}. \end{aligned}$$

17. Since $f(t) = \sin t[h(t) - h(t-2\pi)] = \sin t - \sin t\, h(t-2\pi)$,

$$F(s) = \mathcal{L}\{\sin t - \sin t\, h(t-2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin t\}$$
$$= \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} = \frac{1 - e^{-2\pi s}}{s^2+1}.$$

18. Since $f(t) = \sin t\, h(t-2\pi)$,

$$F(s) = \mathcal{L}\{\sin t\, h(t-2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}.$$

19. Since $f(t) = [h(t) - h(t-\pi)] + \sin t\, h(t-\pi) = 1 - h(t-\pi) + \sin t\, h(t-\pi)$,

$$F(s) = \mathcal{L}\{1 - h(t-\pi) + \sin t\, h(t-\pi)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$$
$$= \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\mathcal{L}\{-\sin t\} = \frac{1 - e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2+1}.$$

20. Since $f(t) = 2e^{-t}[h(t) - h(t-\ln 2)] + h(t-\ln 2) = 2e^{-t} + (1-2e^{-t})h(t-\ln 2)$,

$$F(s) = \mathcal{L}\{2e^{-t} + (1-2e^{-t})h(t-\ln 2)\} = \frac{2}{s+1} + e^{-s\ln 2}\mathcal{L}\{1 - 2e^{-(t+\ln 2)}\}$$
$$= \frac{2}{s+1} + e^{-s\ln 2}\mathcal{L}\{1 - e^{-t}\} = \frac{2}{s+1} + e^{-s\ln 2}\left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{2}{s+1} + \frac{e^{-s\ln 2}}{s(s+1)}.$$

21. $\mathcal{L}\{t^3e^{-5t}\} = \mathcal{L}\{t^3\}_{|s\to s+5} = \left(\frac{3!}{s^4}\right)_{|s\to s+5} = \frac{6}{(s+5)^4}$

22. $\mathcal{L}\{t^2e^{3t}\} = \mathcal{L}\{t^2\}_{|s\to s-3} = \left(\frac{2}{s^3}\right)_{|s\to s-3} = \frac{2}{(s-3)^3}$

23. $\mathcal{L}\{4te^{-t} - 2e^{-3t}\} = 4\mathcal{L}\{t\}_{|s\to s+1} - 2\mathcal{L}\{e^{-3t}\} = 4\left(\frac{1}{s^2}\right)_{|s\to s+1} - \frac{2}{s+3} = \frac{4}{(s+1)^2} - \frac{2}{s+3}$

24. $\mathcal{L}\{5e^{at} - 5e^{-at}\} = \frac{5}{s-a} - \frac{5}{s+a} = \frac{10a}{s^2-a^2}$

25. $\mathcal{L}\{e^t\sin 2t + e^{-t}\cos t\} = \mathcal{L}\{\sin 2t\}_{|s\to s-1} + \mathcal{L}\{\cos t\}_{|s\to s+1} = \left(\frac{2}{s^2+4}\right)_{|s\to s-1} + \left(\frac{s}{s^2+1}\right)_{|s\to s+1}$

$$= \frac{2}{(s-1)^2+4} + \frac{s+1}{(s+1)^2+1}$$

26. $\mathcal{L}\{2e^{-3t}\sin 3t + 4e^{3t}\cos 3t\} = 2\mathcal{L}\{\sin 3t\}_{|s\to s+3} + 4\mathcal{L}\{\cos 3t\}_{|s\to s-3}$

$$= 2\left(\frac{3}{s^2+9}\right)_{|s\to s+3} + 4\left(\frac{s}{s^2+9}\right)_{|s\to s-3} = \frac{6}{(s+3)^2+9} + \frac{4(s-3)}{(s-3)^2+9}$$

27. $\mathcal{L}\{te^t\cos 2t\} = \mathcal{L}\{t\cos 2t\}_{|s\to s-1} = \left[\frac{s^2-4}{(s^2+4)^2}\right]_{|s\to s-1} = \frac{(s-1)^2-4}{[(s-1)^2+4]^2} = \frac{s^2-2s-3}{(s^2-2s+5)^2}$

28. $\mathcal{L}\{te^{-2t}\sin t\} = \mathcal{L}\{t\sin t\}_{|s\to s+2} = \left[\frac{2s}{(s^2+1)^2}\right]_{|s\to s+2} = \frac{2(s+2)}{[(s+2)^2+1]^2} = \frac{2(s+2)}{(s^2+4s+5)^2}$

29. $\mathcal{L}\{2e^t(\cos t + \sin t)\} = 2\mathcal{L}\{\cos t + \sin t\}_{|s\to s-1} = 2\left(\frac{s}{s^2+1} + \frac{1}{s^2+1}\right)_{|s\to s-1}$

$$= 2\left[\frac{(s-1)+1}{(s-1)^2+1}\right] = \frac{2s}{s^2-2s+2}$$

30. $\mathcal{L}\{(t-1)e^{2-3t}\sin 4t\} = e^2\mathcal{L}\{t\sin 4t - \sin 4t\}_{|s\to s+3} = e^2\left[\frac{8s}{(s^2+16)^2} - \frac{4}{s^2+16}\right]_{|s\to s+3}$

$$= e^2\left[\frac{8s-4s^2-64}{(s^2+16)^2}\right]_{|s\to s+3} = \frac{e^2[8(s+3)-4(s+3)^2-64]}{[(s+3)^2+16]^2} = \frac{-4e^2(s^2+4s+19)}{(s^2+6s+25)^2}$$

31. We calculate

$$\mathcal{L}\{t^2e^{ati}\} = \mathcal{L}\{t^2\}_{|s\to s-ai} = \left(\frac{2}{s^3}\right)_{|s\to s-ai} = \frac{2}{(s-ai)^3}.$$

We display real and imaginary parts of both sides of the equation,

$$\mathcal{L}\{t^2(\cos at + i\sin at)\} = \frac{2(s+ai)^3}{(s-ai)^3(s+ai)^3} = \frac{2(s^3+3as^2i-3a^2s-a^3i)}{(s^2+a^2)^3}.$$

When we take real parts, $\mathcal{L}\{t^2\cos at\} = \frac{2(s^3-3a^2s)}{(s^2+a^2)^3}$.

32. We calculate

$$\mathcal{L}\{t^2e^{ati}\} = \mathcal{L}\{t^2\}_{|s\to s-ai} = \left(\frac{2}{s^3}\right)_{|s\to s-ai} = \frac{2}{(s-ai)^3}.$$

We display real and imaginary parts of both sides of the equation,

$$\mathcal{L}\{t^2(\cos at + i\sin at)\} = \frac{2(s+ai)^3}{(s-ai)^3(s+ai)^3} = \frac{2(s^3+3as^2i-3a^2s-a^3i)}{(s^2+a^2)^3}.$$

When we take imaginary parts, $\mathcal{L}\{t^2\sin at\} = \frac{2(3as^2-a^3)}{(s^2+a^2)^3}$.

33. $\mathcal{L}\{(t-2)^2h(t-2)\} = e^{-2s}\mathcal{L}\{(t+2-2)^2\} = e^{-2s}\mathcal{L}\{t^2\} = \dfrac{2e^{-2s}}{s^3}$

34. $\mathcal{L}\{\sin 3(t-4)h(t-4)\} = e^{-4s}\mathcal{L}\{\sin 3(t+4-4)\} = e^{-4s}\mathcal{L}\{\sin 3t\} = \dfrac{3e^{-4s}}{s^2+9}$

35. $\mathcal{L}\{t\,h(t-1)\} = e^{-s}\mathcal{L}\{t+1\} = e^{-s}\left(\dfrac{1}{s^2}+\dfrac{1}{s}\right) = \dfrac{(s+1)e^{-s}}{s^2}$

36. $\mathcal{L}\{(t+5)h(t-3)\} = e^{-3s}\mathcal{L}\{(t+3+5\} = e^{-3s}\mathcal{L}\{t+8\} = e^{-3s}\left(\dfrac{1}{s^2}+\dfrac{8}{s}\right) = \dfrac{(8s+1)e^{-3s}}{s^2}$

37. $\mathcal{L}\{(t^2+2)h(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2+2\} = e^{-s}\mathcal{L}\{t^2+2t+3\} = e^{-s}\left(\dfrac{2}{s^3}+\dfrac{2}{s^2}+\dfrac{3}{s}\right)$

$$= \frac{(2+2s+3s^2)e^{-s}}{s^3}$$

38. $\mathcal{L}\{\cos t\,h(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\cos(t+\pi)\} = e^{-\pi s}\mathcal{L}\{-\cos t\} = \dfrac{-se^{-\pi s}}{s^2+1}$

39. $\mathcal{L}\{\cos t\,h(t-2)\} = e^{-2s}\mathcal{L}\{\cos(t+2)\} = e^{-2s}\mathcal{L}\{\cos 2\cos t - \sin 2\sin t\}$

$$= e^{-2s}\left(\frac{s\cos 2}{s^2+1} - \frac{\sin 2}{s^2+1}\right) = \frac{e^{-2s}(s\cos 2 - \sin 2)}{s^2+1}$$

40. $\mathcal{L}\{e^t h(t-4)\} = e^{-4s}\mathcal{L}\{e^{t+4}\} = e^4e^{-4s}\mathcal{L}\{e^t\} = \dfrac{e^{4-4s}}{s-1}$

41. $\mathcal{L}\{t^2e^th(t-3)\} = e^{-3s}\mathcal{L}\{(t+3)^2e^{t+3}\} = e^{-3s}e^3\mathcal{L}\{(t^2+6t+9)e^t\} = e^{3-3s}\mathcal{L}\{t^2+6t+9\}_{|s\to s-1}$

$$= e^{3-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)_{|s\to s-1} = e^{3-3s}\left[\frac{2}{(s-1)^3}+\frac{6}{(s-1)^2}+\frac{9}{s-1}\right]$$

$$= \frac{(9s^2-12s+5)e^{3-3s}}{(s-1)^3}$$

42. $\mathcal{L}\{e^t\cos 2t\,h(t-1)\} = e^{-s}\mathcal{L}\{e^{t+1}\cos 2(t+1)\} = e^{-s}e\mathcal{L}\{\cos 2(t+1)\}_{|s\to s-1}$

$$= e^{1-s}\mathcal{L}\{\cos 2\cos 2t - \sin 2\sin 2t\}_{|s\to s-1} = e^{1-s}\left(\frac{s\cos 2}{s^2+4} - \frac{2\sin 2}{s^2+4}\right)_{|s\to s-1}$$

$$= e^{1-s}\left[\frac{(s-1)\cos 2}{(s-1)^2+4} - \frac{2\sin 2}{(s-1)^2+4}\right] = \frac{e^{1-s}[s\cos 2 - (\cos 2 + 2\sin 2)]}{s^2-2s+5}$$

43. $F(s) = \dfrac{1}{1-e^{-as}}\mathcal{L}\{t[h(t)-h(t-a)]\} = \dfrac{1}{1-e^{-as}}\mathcal{L}\{t - t\,h(t-a)\} = \dfrac{1}{1-e^{-as}}\left[\dfrac{1}{s^2} - e^{-as}\mathcal{L}\{t+a\}\right]$

$$= \frac{1}{1-e^{-as}}\left[\frac{1}{s^2} - e^{-as}\left(\frac{1}{s^2}+\frac{a}{s}\right)\right] = \frac{1}{1-e^{-as}}\left[\frac{1-e^{-as}}{s^2} - \frac{ae^{-as}}{s}\right] = \frac{1}{s^2} - \frac{ae^{-as}}{s(1-e^{-as})}$$

44. $F(s) = \dfrac{1}{1-e^{-2as}}\mathcal{L}\{[h(t)-h(t-a)] - [h(t-a)-h(t-2a)]\} = \dfrac{1}{1-e^{-2as}}\mathcal{L}\{1 - 2h(t-a) + h(t-2a)\}$

$$= \frac{1}{1-e^{-2as}}\left(\frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s}\right) = \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$

45. $F(s) = \dfrac{1}{1-e^{-\pi s/a}}\mathcal{L}\{\sin at[h(t)-h(t-\pi/a)]\} = \dfrac{1}{1-e^{-\pi s/a}}\mathcal{L}\{\sin at - \sin at\,h(t-\pi/a)\}$

$$= \frac{1}{1-e^{-\pi s/a}}\left[\frac{a}{s^2+a^2} - e^{-\pi s/a}\mathcal{L}\{\sin a(t+\pi/a)\}\right] = \frac{1}{1-e^{-\pi s/a}}\left[\frac{a}{s^2+a^2} - e^{-\pi s/a}\mathcal{L}\{-\sin at\}\right]$$

$$= \frac{1}{1-e^{-\pi s/a}}\left[\frac{a}{s^2+a^2} + \frac{ae^{-\pi s/a}}{s^2+a^2}\right] = \frac{a(1+e^{-\pi s/a})}{(s^2+a^2)(1-e^{-\pi s/a})}$$

46. $F(s) = \dfrac{1}{1-e^{-2as}}\mathcal{L}\{t[h(t)-h(t-a)] + (2a-t)[h(t-a)-h(t-2a)]\}$

$$= \frac{1}{1-e^{-2as}}\mathcal{L}\{t + (2a-2t)h(t-a) + (t-2a)h(t-2a)\}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2}+e^{-as}\mathcal{L}\{2a-2(t+a)\}+e^{-2as}\mathcal{L}\{t+2a-2a\}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2}-\frac{2e^{-as}}{s^2}+\frac{e^{-2as}}{s^2}\right] = \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

47. $F(s) = \frac{1}{1-e^{-2as}}\mathcal{L}\{h(t)-h(t-a)\} = \frac{1}{1-e^{-2as}}\mathcal{L}\{1-h(t-a)\} = \frac{1}{1-e^{-2as}}\left(\frac{1}{s}-\frac{e^{-as}}{s}\right)$

$$= \frac{1-e^{-as}}{s(1+e^{-as})(1-e^{-as})} = \frac{1}{s(1+e^{-as})}$$

48. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}e^t\sin 2t$

49. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2-3}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-3}\right\}$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{(\sqrt{3}+2)/(2\sqrt{3})}{s+\sqrt{3}}+\frac{(\sqrt{3}-2)/(2\sqrt{3})}{s-\sqrt{3}}\right\}$$

$$= e^{-2t}\left[\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)e^{-\sqrt{3}t}+\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)e^{\sqrt{3}t}\right]$$

$$= \left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)e^{-(2+\sqrt{3})t}+\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)e^{(-2+\sqrt{3})t}$$

50. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}_{|t\to t-2} h(t-2) = \{t\}_{|t\to t-2}\, h(t-2) = (t-2)h(t-2)$

51. $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{|t\to t-3} h(t-3) = \{\sin t\}_{|t\to t-3}h(t-3) = \sin(t-3)\,h(t-3)$

52. $\mathcal{L}^{-1}\left\{\frac{se^{-5s}}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}_{|t\to t-5} h(t-5) = \{\cos\sqrt{2}t\}_{|t\to t-5}h(t-5) = \cos\sqrt{2}(t-5)\,h(t-5)$

53. $\mathcal{L}^{-1}\left\{\frac{se^{-s}}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}_{|t\to t-1} h(t-1) = \left\{\frac{t}{4}\sin 2t\right\}_{|t\to t-1} h(t-1)$

$$= \frac{1}{4}(t-1)\sin 2(t-1)\,h(t-1)$$

54. $\mathcal{L}^{-1}\left\{\frac{1}{4s^2-6s-5}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2-3s/2-5/4}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-3/4)^2-29/16}\right\}$

$$= \frac{1}{4}e^{3t/4}\mathcal{L}^{-1}\left\{\frac{1}{s^2-29/16}\right\} = \frac{1}{4}e^{3t/4}\mathcal{L}^{-1}\left\{\frac{-2/\sqrt{29}}{s+\sqrt{29}/4}+\frac{2/\sqrt{29}}{s-\sqrt{29}/4}\right\}$$

$$= \frac{1}{2\sqrt{29}}e^{3t/4}(-e^{-\sqrt{29}t/4}+e^{\sqrt{29}t/4}) = \frac{\sqrt{29}}{58}[e^{(3+\sqrt{29})t/4}-e^{(3-\sqrt{29})t/4}]$$

55. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-2}-\frac{1}{s-1}\right\} = 2e^{2t}-e^t$

56. $\mathcal{L}^{-1}\left\{\frac{4s+1}{(s^2+s)(4s^2-1)}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4s+1}{s(s+1)(s+1/2)(s-1/2)}\right\}$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{-\frac{4}{s}+\frac{4}{s+1}-\frac{4}{s+1/2}+\frac{4}{s-1/2}\right\} = -1+e^{-t}-e^{-t/2}+e^{t/2}$$

57. $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}_{|t\to t-3} h(t-3) = \{e^{-5t}\}_{|t\to t-3}\, h(t-3) = e^{-5(t-3)}h(t-3)$

58. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\}_{|t\to t-2} h(t-2) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+2}\right\}_{|t\to t-2} h(t-2)$

$$= \left\{e^{-t} - e^{-2t}\right\}_{|t\to t-2} h(t-2) = [e^{-(t-2)} - e^{-2(t-2)}]h(t-2)$$
$$= [e^{2-t} - e^{2(2-t)}]h(t-2)$$

59. $$\mathcal{L}^{-1}\left\{\frac{1}{s^3+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2-s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/3}{s+1} + \frac{2/3 - s/3}{s^2-s+1}\right\}$$
$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{-(s-1/2)+3/2}{(s-1/2)^2+3/4}\right\} = \frac{1}{3}e^{-t} + \frac{1}{3}e^{t/2}\mathcal{L}^{-1}\left\{\frac{3/2-s}{s^2+3/4}\right\}$$
$$= \frac{1}{3}e^{-t} + \frac{1}{3}e^{t/2}\left(\sqrt{3}\sin\frac{\sqrt{3}t}{2} - \cos\frac{\sqrt{3}t}{2}\right)$$

60. $$\mathcal{L}^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{5s-2}{s^2+4s/3+8/3}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{5(s+2/3)-16/3}{(s+2/3)^2+20/9}\right\}$$
$$= \frac{1}{3}e^{-2t/3}\mathcal{L}^{-1}\left\{\frac{5s-16/3}{s^2+20/9}\right\} = \frac{1}{3}e^{-2t/3}\left(5\cos\frac{2\sqrt{5}t}{3} - \frac{8}{\sqrt{5}}\sin\frac{2\sqrt{5}t}{3}\right)$$

61. Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$,

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}(1-e^{-s})}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}-e^{-2s}}{s(s^2+1)}\right\} = [1-\cos(t-1)]h(t-1) - [1-\cos(t-2)]h(t-2).$$

62. $$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4} - \frac{1}{(s+1)^5}\right\}$$
$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^4} - \frac{1}{s^5}\right\} = e^{-t}\left(\frac{t^3}{3!} - \frac{t^4}{4!}\right) = \frac{t^3(4-t)e^{-t}}{24}$$

63. $$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}\right\}$$
$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1} + \frac{2}{(s+1)^2+4}\right\}$$
$$= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} + \frac{2}{s^2+4}\right\} = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

64. $$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2-4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1/8}{s-2} + \frac{1/4}{(s-2)^2} - \frac{1/8}{s+2} + \frac{1/4}{(s+2)^2}\right\}$$
$$= \frac{1}{8}e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s^2}\right\} + \frac{1}{8}e^{-2t}\mathcal{L}^{-1}\left\{\frac{2}{s^2} - \frac{1}{s}\right\} = \frac{1}{8}e^{2t}(1+2t) + \frac{1}{8}e^{-2t}(2t-1)$$

65. We use geometric series to write $F(s)$ in the form

$$F(s) = \frac{1}{s}\sum_{n=0}^{\infty}(e^{-s})^n = \sum_{n=0}^{\infty}\frac{e^{-ns}}{s}.$$

We can now take inverse transforms,

$$f(t) = \sum_{n=0}^{\infty} h(t-n).$$

Except for values at the integers, this is the floor function $\lfloor t \rfloor$.

66. We use geometric series to write $F(s)$ in the form

$$F(s) = \frac{1}{s}\sum_{n=0}^{\infty}(-e^{-s})^n = \sum_{n=0}^{\infty}(-1)^n\frac{e^{-ns}}{s}.$$

We can now take inverse transforms,

$$f(t)=\sum_{n=0}^{\infty}(-1)^n h(t-n).$$

67. We use geometric series to write $F(s)$ in the form

$$F(s)=\frac{1}{s^2+4}\sum_{n=0}^{\infty}(e^{-3s})^n=\sum_{n=0}^{\infty}\frac{e^{-3ns}}{s^2+4}.$$

We can now take inverse transforms,

$$f(t)=\frac{1}{2}\sum_{n=0}^{\infty}\sin 2(t-3n)h(t-3n).$$

68. We use geometric series to write $F(s)$ in the form

$$F(s)=\left(\frac{1/5}{s}-\frac{s/5}{s^2+5}\right)\sum_{n=0}^{\infty}(e^{-2s})^n=\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{1}{s}-\frac{s}{s^2+5}\right)e^{-2ns}.$$

We can now take inverse transforms,

$$f(t)=\frac{1}{5}\sum_{n=0}^{\infty}\left[1-\cos\sqrt{5}(t-2n)\right]h(t-2n).$$

69. We use geometric series to write $F(s)$ in the form

$$F(s)=\left(\frac{1/2}{s^2}+\frac{1/2}{s^2+2}\right)e^{-2s}\sum_{n=0}^{\infty}(-e^{-s})^n=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{s^2}+\frac{1}{s^2+2}\right)e^{-(n+2)s}.$$

We can now take inverse transforms,

$$f(t)=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[(t-n-2)+\frac{1}{\sqrt{2}}\sin\sqrt{2}(t-n-2)\right]h(t-n-2).$$

70. (a) Using property 6.8b,

$$f(t)=\mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+4}\right\}=e^{-t}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}=e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right).$$

(b) Since $s^2+2s+5=0$ has roots $s=\dfrac{-2\pm\sqrt{4-20}}{2}=-1\pm 2i$, the quadratic factors into $(s+1-2i)(s+1+2i)$, and the partial fraction decomposition of $F(s)$ is of the form

$$\frac{s+2}{s^2+2s+5}=\frac{A}{s+1-2i}+\frac{B}{s+1+2i}.$$

Constants A and B can be obtained with the "cover up" method in Appendix D,

$$A=\frac{-1+2i+2}{4i}=\frac{2-i}{4},\qquad B=\frac{-1-2i+2}{-4i}=\frac{2+i}{4}.$$

Thus,

$$\frac{s+2}{s^2+2s+5}=\frac{1}{4}\left(\frac{2-i}{s+1-2i}+\frac{2+i}{s+1+2i}\right),$$

and

$$\begin{aligned}f(t)&=\frac{1}{4}\left[(2-i)e^{(-1+2i)t}+(2+i)e^{-(1+2i)t}\right]\\&=\frac{e^{-t}}{4}\left[(2-i)(\cos 2t+i\sin 2t)+(2+i)(\cos 2t-i\sin 2t)\right]=\frac{e^{-t}}{4}(4\cos 2t+2\sin 2t).\end{aligned}$$

71. The given property can also be written in terms of inverse transforms as

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\mathcal{L}^{-1}\left\{F(s)\right\}.$$

When we apply this to

$$\frac{d}{ds}\left[\frac{1}{(s^2+a^2)^n}\right] = \frac{-2ns}{(s^2+a^2)^{n+1}},$$

we get

$$\mathcal{L}^{-1}\left\{\frac{-2ns}{(s^2+a^2)^{n+1}}\right\} = -t\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^n}\right\}.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^{n+1}}\right\} = \frac{t}{2n}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^n}\right\}.$$

To prove the second formula, we begin with

$$\begin{aligned}\frac{d}{ds}\left[\frac{s}{(s^2+a^2)^n}\right] &= \frac{1}{(s^2+a^2)^n} - \frac{2ns^2}{(s^2+a^2)^{n+1}} = \frac{1}{(s^2+a^2)^n} - \frac{2n(s^2+a^2)-2na^2}{(s^2+a^2)^{n+1}}\\ &= \frac{1}{(s^2+a^2)^n} - \frac{2n}{(s^2+a^2)^n} + \frac{2na^2}{(s^2+a^2)^{n+1}}.\end{aligned}$$

If we now take inverse transforms,

$$-t\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^n}\right\} = (1-2n)\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^n}\right\} + 2na^2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^{n+1}}\right\}.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^{n+1}}\right\} = \frac{-t}{2na^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^n}\right\} + \frac{2n-1}{2na^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^n}\right\}.$$

72. For line 5 in the table,

$$\mathcal{L}^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\} = 2a\left[\frac{t}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right] = at\left(\frac{1}{a}\sin at\right) = t\sin at.$$

For line 6,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} - 2a^2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}\\ &= \frac{1}{a}\sin at - 2a^2\left[\frac{-t}{2a^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + \frac{1}{2a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]\\ &= \frac{1}{a}\sin at + t\cos at - \frac{1}{a}\sin at = t\cos at.\end{aligned}$$

For line 7,

$$\mathcal{L}^{-1}\left\{\frac{2a^3}{(s^2+a^2)^2}\right\} = 2a^3\left[\frac{-t}{2a^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + \frac{1}{2a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right] = -at\cos at + \sin at,$$

For line 8,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2as^2}{(s^2+a^2)^2}\right\} &= 2a\mathcal{L}^{-1}\left\{\frac{(s^2+a^2)-a^2}{(s^2+a^2)^2}\right\} = 2a\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} - 2a^3\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}\\ &= 2a\left(\frac{1}{a}\sin at\right) - 2a^3\left[\frac{-t}{2a^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + \frac{1}{2a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]\\ &= 2\sin at + at\cos at - \sin at = \sin at + at\cos at.\end{aligned}$$

73. Using the second reduction of order formula in Exercise 71,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^3}\right\} = \frac{-t}{4a^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} + \frac{3}{4a^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}.$$

We can now use Table 6.2,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^3}\right\} &= -\frac{t}{4a^2}\left[\frac{t}{2a}\sin at\right] + \frac{3}{4a^2}\left[\frac{1}{2a^3}(\sin at - at\cos at)\right]\\ &= \frac{-t^2}{8a^3}\sin at + \frac{3}{8a^5}(\sin at - at\cos at).\end{aligned}$$

74. Using the first reduction of order formula and Table 6.2,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^3}\right\} = \frac{t}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} + \frac{t}{4}\left[\frac{1}{2a^3}(\sin at - at\cos at)\right] = \frac{t}{8a^3}(\sin at - at\cos at).$$

75. When we complete the square,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+5)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{[(s-1)^2+4]^3}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^3}\right\}.$$

We now use the second reduction of order formula and Table 6.2,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+5)^3}\right\} &= e^t\left[\frac{-t}{16}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} + \frac{3}{16}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}\right]\\ &= e^t\left[\frac{-t}{16}\left(\frac{t}{4}\sin 2t\right) + \frac{3}{16}\left(\frac{-t}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right)\right]\\ &= e^t\left[\frac{-t^2}{64}\sin 2t - \frac{3t}{128}\cos 2t + \frac{3}{256}\sin 2t\right]\\ &= \frac{e^t}{256}\left[(3-4t^2)\sin 2t - 6t\cos 2t\right].\end{aligned}$$

76. When we complete the square,

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2-4s+13)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-2)+4}{[(s-2)^2+9]^3}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{s+4}{(s^2+9)^3}\right\}.$$

We now use the reduction of order formulas and Table 6.2,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+5)^3}\right\} &= e^{2t}\left[\frac{t}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} - \frac{4t}{36}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} + \frac{4(3)}{36}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\}\right]\\ &= e^{2t}\left[\frac{t}{4}\left(\frac{1}{54}\right)(\sin 3t - 3t\cos 3t) - \frac{t}{9}\left(\frac{1}{6}\right)t\sin 3t\right.\\ &\qquad\left.+\frac{1}{3}\left(\frac{1}{54}\right)(\sin 3t - 3\cos 3t)\right]\\ &= \frac{e^{2t}}{648}[(4+3t-12t^2)\sin 3t - (12t+9t^2)\cos 3t].\end{aligned}$$

77. If $F(s) = \mathcal{L}\{f(t)\}$ for $s > \alpha$, then presumably, $f(t)$ is $O(e^{\alpha t})$. This means that $e^{at}f(t)$ is $O(e^{(\alpha+a)t})$. Thus, $F(s-a)$ is the transform of $e^{at}f(t)$ if $s > \alpha + a$.

78.

$$\begin{aligned}F(s) &= \frac{1}{1-e^{-4s}}\mathcal{L}\left\{\frac{t^2}{4}[h(t)-h(t-1)] - \frac{1}{4}(t^2-4t+2)[h(t-1)-h(t-3)]\right.\\ &\qquad\left.+\frac{1}{4}(t-4)^2[h(t-3)-h(t-4)]\right\}\\ &= \frac{1}{1-e^{-4s}}\mathcal{L}\left\{\frac{t^2}{4} - \frac{1}{2}(t^2-2t+1)h(t-1) + \frac{1}{2}(t^2-6t+9)h(t-3) - \frac{1}{4}(t-4)^2h(t-4)\right\}\end{aligned}$$

$$= \frac{1}{4(1-e^{-4s})}\left[\frac{2}{s^3} - 2e^{-s}\mathcal{L}\{(t+1)^2 - 2(t+1)+1\} + 2e^{-3s}\mathcal{L}\{(t+3)^2 - 6(t+3)+9\} \right.$$
$$\left. -e^{-4s}\mathcal{L}\{(t+4-4)^2\}\right]$$
$$= \frac{1}{4(1-e^{-4s})}\left[\frac{2}{s^3} - 2e^{-s}\mathcal{L}\{t^2\} + 2e^{-3s}\mathcal{L}\{t^2\} - e^{-4s}\mathcal{L}\{t^2\}\right]$$
$$= \frac{1}{4(1-e^{-4s})}\left[\frac{2}{s^3} + \frac{2}{s^3}(-2e^{-s} + 2e^{-3s} - e^{-4s})\right] = \frac{1}{2s^3(1-e^{-4s})}(1 - 2e^{-s} + 2e^{-3s} - e^{-4s})$$
$$= \frac{(1-e^{-s})^2(1-e^{-2s})}{2s^3(1+e^{-2s})(1-e^{-2s})} = \frac{(1-e^{-s})^2}{2s^3(1+e^{-2s})}$$

79. If we set $u = at$ in $\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}\,dt$, we get

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(u)e^{-s(u/a)}\left(\frac{du}{a}\right) = \frac{1}{a}\int_0^\infty f(u)e^{-(s/a)u}\,du = \frac{1}{a}F\left(\frac{s}{a}\right).$$

Since $F(s)$ is valid for $s > a$, it follows that $\mathcal{L}\{f(at)\}$ is valid for $s/a > \alpha$, which implies that $s > \alpha a$.

EXERCISES 6.3

1. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 0] + 3[sY - 1] - 4Y = \frac{1}{s^2} + \frac{3}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{3s+1}{s^2(s^2+3s-4)} + \frac{s+3}{s^2+3s-4}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3s+1}{s^2(s^2+3s-4)} + \frac{s+3}{s^2+3s-4}\right\} = \mathcal{L}^{-1}\left\{-\frac{15/16}{s} - \frac{1/4}{s^2} + \frac{27/80}{s+4} + \frac{8/5}{s-1}\right\}$$
$$= -\frac{15}{16} - \frac{t}{4} + \frac{27}{80}e^{-4t} + \frac{8}{5}e^t.$$

2. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 2[sY - 1] - Y = \frac{1}{s-1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{(s-1)(s^2+2s-1)} + \frac{s+4}{s^2+2s-1}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2+2s-1)} + \frac{s+4}{s^2+2s-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{s/2+5/2}{s^2+2s-1}\right\}$$
$$= \frac{1}{2}\left[e^t + \mathcal{L}^{-1}\left\{\frac{s+5}{s^2+2s-1}\right\}\right] = \frac{1}{2}\left[e^t + \mathcal{L}^{-1}\left\{\frac{(s+1)+4}{(s+1)^2-2}\right\}\right]$$
$$= \frac{1}{2}\left[e^t + e^{-t}\mathcal{L}^{-1}\left\{\frac{s+4}{s^2-2}\right\}\right] = \frac{1}{2}e^t + \frac{1}{2}e^{-t}\mathcal{L}^{-1}\left\{\frac{-\sqrt{2}+1/2}{s+\sqrt{2}} + \frac{\sqrt{2}+1/2}{s-\sqrt{2}}\right\}$$
$$= \frac{1}{2}e^t + \frac{1}{2}e^{-t}\left[\left(\frac{1}{2}-\sqrt{2}\right)e^{-\sqrt{2}t} + \left(\frac{1}{2}+\sqrt{2}\right)e^{\sqrt{2}t}\right]$$
$$= \frac{1}{2}e^t + \left(\frac{1}{4} - \frac{\sqrt{2}}{2}\right)e^{-(1+\sqrt{2})t} + \left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right)e^{(-1+\sqrt{2})t}.$$

3. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + Y = \frac{2}{s+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2}{(s+1)(s^2+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1-s}{s^2+1}\right\} = e^{-t} - \cos t + \sin t.$$

4. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(0) - 1] + 2[sY] + Y = \frac{1}{s^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2+2s+1} + \frac{1}{s^2(s^2+2s+1)} = \frac{1}{(s+1)^2} + \frac{1}{s^2(s+1)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} + \frac{1}{s^2(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2} - \frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1}\right\} \\ &= -2 + t + 2e^{-t} + e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = t - 2 + 2e^{-t} + 2te^{-t}. \end{aligned}$$

5. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1)] - 2[sY - 1] + Y = \frac{2}{(s-1)^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-2}{s^2-2s+1} + \frac{2}{(s-1)^3(s^2-2s+1)} = \frac{s-2}{(s-1)^2} + \frac{2}{(s-1)^5}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s-2}{(s-1)^2} + \frac{2}{(s-1)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-1)-1}{(s-1)^2} + \frac{2}{(s-1)^5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{2}{(s-1)^5}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^5}\right\} \\ &= e^t\left(1 - t + \frac{2t^4}{4!}\right) = e^t\left(1 - t + \frac{t^4}{12}\right). \end{aligned}$$

6. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) + 2] + Y = \frac{1}{s^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s-3}{s^2+1}\right\} = t + \cos t - 3\sin t.$$

7. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 1] + 2[sY] + 5Y = \frac{1}{(s+1)^2+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{2/3}{s^2+2s+5} + \frac{1/3}{s^2+2s+2}\right\} \\
&= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \frac{2}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} + \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= \frac{1}{3}e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t.
\end{aligned}$$

8. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 2s - 1] + 6[sY - 2] + Y = \frac{3}{s^2+9}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s+13}{s^2+6s+1} + \frac{3}{(s^2+9)(s^2+6s+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2s+13}{s^2+6s+1} + \frac{3}{(s^2+9)(s^2+6s+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{-\frac{9s/194 + 12/194}{s^2+9} + \frac{397s/194 + 2588/194}{s^2+6s+1}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{1}{194}\mathcal{L}^{-1}\left\{\frac{397(s+3)+1397}{(s+3)^2-8}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{e^{-3t}}{194}\mathcal{L}^{-1}\left\{\frac{397s+1397}{s^2-8}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{e^{-3t}}{194}\mathcal{L}^{-1}\left\{\frac{(-1397+794\sqrt{2})/(4\sqrt{2})}{s+2\sqrt{2}} + \frac{(794\sqrt{2}+1397)/(4\sqrt{2})}{s-2\sqrt{2}}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{e^{-3t}}{776\sqrt{2}}\left[(-1397+794\sqrt{2})e^{-2\sqrt{2}t} + (1397+794\sqrt{2})e^{2\sqrt{2}t}\right] \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{1}{776\sqrt{2}}\left[(-1397+794\sqrt{2})e^{-(3+2\sqrt{2})t} + (1397+794\sqrt{2})e^{(-3+2\sqrt{2})t}\right].
\end{aligned}$$

9. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) + 2] + [sY - 1] - 6Y = \frac{1}{s^2} + \frac{s}{s^2+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-1}{s^2+s-6} + \frac{1}{s^2(s^2+s-6)} + \frac{s}{(s^2+1)(s^2+s-6)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s-1}{s^2+s-6} + \frac{1}{s^2(s^2+s-6)} + \frac{s}{(s^2+1)(s^2+s-6)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s-1}{(s+3)(s-2)} + \frac{1}{s^2(s+3)(s-2)} + \frac{s}{(s^2+1)(s+3)(s-2)}\right\}
\end{aligned}$$

$$= \mathcal{L}^{-1}\left\{\frac{-1/36}{s} - \frac{1/6}{s^2} + \frac{377/450}{s+3} + \frac{33/100}{s-2} + \frac{-7s/50+1/50}{s^2+1}\right\}$$
$$= -\frac{1}{36} - \frac{t}{6} + \frac{377}{450}e^{-3t} + \frac{33}{100}e^{2t} - \frac{7}{50}\cos t + \frac{1}{50}\sin t.$$

10. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(-1) - 2] - 4[sY - (-1)] + 5Y = \frac{1}{(s+3)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{-s+6}{s^2-4s+5} + \frac{1}{(s+3)^2(s^2-4s+5)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-s+6}{s^2-4s+5} + \frac{1}{(s+3)^2(s^2-4s+5)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{5/338}{s+3} + \frac{1/26}{(s+3)^2} + \frac{-343s/338 + 2050/338}{s^2-4s+5}\right\}$$
$$= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{1}{338}\mathcal{L}^{-1}\left\{\frac{-343(s-2)+1364}{(s-2)^2+1}\right\}$$
$$= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{e^{2t}}{338}\mathcal{L}^{-1}\left\{\frac{-343s+1364}{s^2+1}\right\}$$
$$= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{e^{2t}}{338}(-343\cos t + 1364\sin t).$$

11. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 1] + 4Y = \mathcal{L}\{h(t) - h(t-1)\} = \frac{1}{s} - \frac{e^{-s}}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2+4} + \frac{1-e^{-s}}{s(s^2+4)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4} + \frac{1-e^{-s}}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4} + (1-e^{-s})\left(\frac{1/4}{s} - \frac{s/4}{s^2+4}\right)\right\}$$
$$= \frac{1}{2}\sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]h(t-1).$$

12. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] - 4Y = \mathcal{L}\{\cos^2 t\} = \mathcal{L}\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2s} + \frac{s}{2(s^2+4)}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{2s(s^2+2s-4)} + \frac{s}{2(s^2+2s-4)(s^2+4)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2s(s^2+2s-4)} + \frac{s}{2(s^2+2s-4)(s^2+4)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{-1/8}{s} + \frac{-s/20+1/20}{s^2+4} + \frac{7s/40+12/40}{s^2+2s-4}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{1}{40}\mathcal{L}^{-1}\left\{\frac{7(s+1)+5}{(s+1)^2-5}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{e^{-t}}{40}\mathcal{L}^{-1}\left\{\frac{7s+5}{s^2-5}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{e^{-t}}{40}\mathcal{L}^{-1}\left\{\frac{(7\sqrt{5}-5)/(2\sqrt{5})}{s+\sqrt{5}} + \frac{(7\sqrt{5}+5)/(2\sqrt{5})}{s-\sqrt{5}}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{e^{-t}}{80\sqrt{5}}[(7\sqrt{5}-5)e^{-\sqrt{5}t} + (7\sqrt{5}+5)e^{\sqrt{5}t}]\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{1}{80}[(7-\sqrt{5})e^{-(1+\sqrt{5})t} + (7+\sqrt{5})e^{(-1+\sqrt{5})t}].
\end{aligned}$$

13. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 2] - 3[sY] + 2Y = \frac{16}{s^3} + \frac{12}{s+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2}{s^2-3s+2} + \frac{16}{s^3(s^2-3s+2)} + \frac{12}{(s+1)(s^2-3s+2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s^2-3s+2} + \frac{16}{s^3(s^2-3s+2)} + \frac{12}{(s+1)(s^2-3s+2)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s-1)} + \frac{16}{s^3(s-2)(s-1)} + \frac{12}{(s+1)(s-2)(s-1)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{14}{s} + \frac{12}{s^2} + \frac{8}{s^3} + \frac{2}{s+1} + \frac{8}{s-2} - \frac{24}{s-1}\right\}\\
&= 14 + 12t + 4t^2 + 2e^{-t} + 8e^{2t} - 24e^{t}.
\end{aligned}$$

14. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 4[sY] - 2Y = \frac{4}{s^2+16}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{4}{(s^2+16)(s^2+4s-2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{4}{(s^2+16)(s^2+4s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-4s/145 - 18/145}{s^2+16} + \frac{4s/145+34/145}{s^2+4s-2}\right\}\\
&= \frac{1}{145}\left[-4\cos 4t - \frac{9}{2}\sin 4t + \mathcal{L}^{-1}\left\{\frac{4(s+2)+26}{(s+2)^2-6}\right\}\right]
\end{aligned}$$

$$= \frac{1}{290}\left[-8\cos 4t - 9\sin 4t + 2e^{-2t}\mathcal{L}^{-1}\left\{\frac{4s+26}{s^2-6}\right\}\right]$$
$$= \frac{1}{290}\left[-8\cos 4t - 9\sin 4t + 4e^{-2t}\mathcal{L}^{-1}\left\{\frac{(2\sqrt{6}-13)/(2\sqrt{6})}{s+\sqrt{6}} + \frac{(2\sqrt{6}+13)/(2\sqrt{6})}{s-\sqrt{6}}\right\}\right]$$
$$= \frac{1}{290}\left\{-8\cos 4t - 9\sin 4t + 4e^{-2t}\left[\left(1-\frac{13}{2\sqrt{6}}\right)e^{-\sqrt{6}t} + \left(1+\frac{13}{2\sqrt{6}}\right)e^{\sqrt{6}t}\right]\right\}$$
$$= -\frac{1}{290}(8\cos 4t + 9\sin 4t) + \frac{\sqrt{6}}{870}\left[(2\sqrt{6}-13)e^{-(2+\sqrt{6})t} + (2\sqrt{6}+13)e^{(-2+\sqrt{6})t}\right].$$

15. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 1] + 8[sY] + 41Y = \frac{1}{(s+2)^2+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2+8s+41} + \frac{1}{(s^2+4s+5)(s^2+8s+41)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+8s+41} + \frac{1}{(s^2+4s+5)(s^2+8s+41)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{-s/200+5/200}{s^2+4s+5} + \frac{s/200+199/200}{s^2+8s+41}\right\}$$
$$= \frac{1}{200}\mathcal{L}^{-1}\left\{\frac{-(s+2)+7}{(s+2)^2+1} + \frac{(s+4)+195}{(s+4)^2+25}\right\}$$
$$= \frac{1}{200}\left[e^{-2t}\mathcal{L}^{-1}\left\{\frac{-s+7}{s^2+1}\right\} + e^{-4t}\mathcal{L}^{-1}\left\{\frac{s+195}{s^2+25}\right\}\right]$$
$$= \frac{1}{200}\left[e^{-2t}(-\cos t + 7\sin t) + e^{-4t}(\cos 5t + 39\sin 5t)\right].$$

16. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] + Y = \mathcal{L}\{t[h(t) - h(t-1)]\} = \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2(s^2+2s+1)} - \frac{e^{-s}(s+1)}{s^2(s^2+2s+1)} = \frac{1}{s^2(s+1)^2} - \frac{e^{-s}}{s^2(s+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2} - \frac{e^{-s}}{s^2(s+1)}\right\}$$
$$= \mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2} - e^{-s}\left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)\right\}$$
$$= -2 + t + 2e^{-t} + te^{-t} + [1 - (t-1) - e^{-(t-1)}]h(t-1)$$
$$= -2 + t + 2e^{-t} + te^{-t} + [2 - t - e^{-(t-1)}]h(t-1).$$

17. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - A] + 9Y = \frac{s}{s^2+4}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+A}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+A}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{s/5}{s^2+4} + \frac{4s/5+A}{s^2+9}\right\} \\ &= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t. \end{aligned}$$

Since $y(\pi/2) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3} \quad \Longrightarrow \quad A = \frac{12}{5}.$$

Thus, $y(t) = \dfrac{1}{5}\cos 2t + \dfrac{4}{5}\cos 3t + \dfrac{4}{5}\sin 3t.$

18. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - A] + 3[sY - 1] - 4Y = \frac{2}{s+4}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+A+3}{s^2+3s-4} + \frac{2}{(s+4)(s^2+3s-4)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+A+3}{s^2+3s-4} + \frac{2}{(s+4)(s^2+3s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+A+3}{(s+4)(s-1)} + \frac{2}{(s+4)^2(s-1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{5A/25+22/25}{s-1} + \frac{(3-5A)/25}{s+4} - \frac{2/5}{(s+4)^2}\right\} \\ &= \left(\frac{A}{5} + \frac{22}{25}\right)e^t + \left(\frac{3}{25} - \frac{A}{5}\right)e^{-4t} - \frac{2t}{5}e^{-4t}. \end{aligned}$$

Since $y(1) = 1$,

$$1 = \left(\frac{A}{5} + \frac{22}{5}\right)e + \left(\frac{3}{25} - \frac{A}{5}\right)e^{-4} - \frac{2}{5}e^{-4} \quad \Longrightarrow \quad A = \frac{25e^4 - 22e^5 + 7}{5(e^5-1)}.$$

Thus,

$$\begin{aligned} y(t) &= \left[\frac{25e^4 - 22e^5 + 7}{25(e^5-1)} + \frac{22}{25}\right]e^t + \left[\frac{3}{25} - \frac{25e^4 - 22e^5 + 7}{25(e^5-1)}\right]e^{-4t} - \frac{2t}{5}e^{-4t} \\ &= \left(\frac{5e^4 - 3}{5e^5 - 5}\right)e^t + \left(\frac{5e^5 - 5e^4 - 2}{5e^5 - 5}\right)e^{-4t} - \frac{2t}{5}e^{-4t}. \end{aligned}$$

19. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - A] + 2[sY] + 5Y = \frac{1}{(s+1)^2+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{A}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2} + \frac{A-1/3}{s^2+2s+5}\right\} \\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \left(A - \frac{1}{3}\right)\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} \\ &= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \left(A-\frac{1}{3}\right)e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &= \frac{1}{3}e^{-t}\sin t + \frac{1}{2}\left(A - \frac{1}{3}\right)e^{-t}\sin 2t. \end{aligned}$$

Since $y(\pi/4) = 1$,

$$1 = \frac{1}{3}e^{-\pi/4}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(A - \frac{1}{3}\right)e^{-\pi/4} \quad\Longrightarrow\quad A = \frac{1}{3} - \frac{\sqrt{2}}{3} + 2e^{\pi/4}.$$

Thus, $y(t) = \dfrac{1}{3}e^{-t}\sin t + \left(e^{\pi/4} - \dfrac{1}{3\sqrt{2}}\right)e^{-t}\sin 2t.$

20. (a) If we let $y(0) = A$ and $y'(0) = B$, and take Laplace transforms of the differential equation,

$$(s^2Y - As - B) + 2(sY - A) - 3Y = \frac{2}{s^2+4} \quad\Longrightarrow\quad Y(s) = \frac{As + (B+2A) + \dfrac{2}{s^2+4}}{s^2+2s-3}.$$

Since the partial fraction decomposition of $\dfrac{2}{(s^2+4)(s^2+2s-3)}$ is

$$\frac{2}{(s^2+4)(s^2+2s-3)} = \frac{1/10}{s-1} - \frac{1/26}{s+3} - \frac{4s/65 + 14/65}{s^2+4},$$

we can write that

$$Y(s) = -\frac{4s/65 + 14/65}{s^2+4} + \frac{C}{s-1} + \frac{D}{s+3}.$$

Inverse transforms give

$$y(t) = -\frac{7}{65}\sin 2t - \frac{4}{65}\cos 2t + Ce^t + De^{-3t}.$$

The initial conditions require

$$3 = \frac{4}{65} + Ce^{\pi/2} + De^{-3\pi/2}, \qquad -1 = \frac{14}{65} + Ce^{\pi/2} - 3De^{-3\pi/2}.$$

The solution of these is $C = (19/10)e^{-\pi/2}$ and $D = (27/26)e^{3\pi/2}$. Thus,

$$y(t) = -\frac{7}{65}\sin 2t - \frac{4}{65}\cos 2t + \frac{19}{10}e^{t-\pi/2} + \frac{27}{26}e^{-3(t-\pi/2)}.$$

(b) If we set $u = t - \pi/2$, the initial-value problem for $y(u)$ is

$$y'' + 2y' - 3y = \sin 2(u + \pi/2) = -\sin 2u, \qquad y(0) = 3, \quad y'(0) = -1.$$

When we take Laplace transforms,

$$(s^2Y - 3s + 1) + 2(sY - 3) - 3Y = \frac{-2}{s^2+4} \quad\Longrightarrow\quad Y(s) = \frac{3s + 5 - \dfrac{2}{s^2+4}}{s^2+2s-3}.$$

Partial fractions give

$$Y(s) = \frac{19/10}{s-1} + \frac{27/26}{s+3} + \frac{14s/65 + 4/65}{s^2+4},$$

and therfore

$$y(u) = \frac{19}{10}e^u + \frac{27}{26}e^{-3u} + \frac{7}{65}\sin 2u + \frac{4}{65}\cos 2u.$$

Returning now to variable t,

$$\begin{aligned} y(t) &= \frac{19}{10}e^{t-\pi/2} + \frac{27}{26}e^{-3(t-\pi/2)} + \frac{7}{65}\sin 2(t-\pi/2) + \frac{4}{65}\cos 2(t-\pi/2) \\ &= \frac{19}{10}e^{t-\pi/2} + \frac{27}{26}e^{-3(t-\pi/2)} - \frac{7}{65}\sin 2t - \frac{4}{65}\cos 2t. \end{aligned}$$

21. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1)] - 4[sY - 1] + 3Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-4}{s^2-4s+3} + \frac{F(s)}{s^2-4s+3} = \frac{s-4}{(s-1)(s-3)} + \frac{F(s)}{(s-1)(s-3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s-4}{(s-1)(s-3)} + \frac{F(s)}{(s-1)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s-1} - \frac{1/2}{s-3} + \left(\frac{-1/2}{s-1} + \frac{1/2}{s-3}\right)F(s)\right\} \\ &= \frac{3}{2}e^t - \frac{1}{2}e^{3t} + \frac{1}{2}\int_0^t [-e^{t-u} + e^{3(t-u)}]f(u)\,du. \end{aligned}$$

22. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 4[sY] + 6Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{F(s)}{s^2+4s+6}.$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \frac{e^{-2t}}{\sqrt{2}}\sin\sqrt{2}t,$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+4s+6}\right\} = \frac{1}{\sqrt{2}}\int_0^t e^{-2(t-u)}\sin\sqrt{2}(t-u)f(u)\,du.$$

23. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 16Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As+B}{s^2+16} + \frac{F(s)}{s^2+16}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = A\cos 4t + \frac{B}{4}\sin 4t + \frac{1}{4}\int_0^t \sin 4(t-u)f(u)\,du$$
$$= A\cos 4t + C\sin 4t + \frac{1}{4}\int_0^t \sin 4(t-u)f(u)\,du.$$

24. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 3[sY - A] + 2Y = F(s-1).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As+B+3A}{s^2+3s+2} + \frac{F(s-1)}{s^2+3s+2} = \frac{As+B+3A}{(s+1)(s+2)} + \frac{F(s-1)}{(s+1)(s+2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{As+B+3A}{(s+1)(s+2)} + \frac{F(s-1)}{(s+1)(s+2)}\right\}$$
$$\mathcal{L}^{-1}\left\{\frac{2A+B}{s+1} - \frac{A+B}{s+2} + \left(\frac{1}{s+1} - \frac{1}{s+2}\right)F(s-1)\right\}$$
$$= (2A+B)e^{-t} - (A+B)e^{-2t} + \int_0^t [e^{-(t-u)} - e^{-2(t-u)}]e^u f(u)\,du$$
$$= Ce^{-t} + De^{-2t} + \int_0^t (e^{2u-t} - e^{3u-2t})f(u)\,du.$$

25. Since $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$,

$$f(t) = \int_0^t e^{-u}\,du = \{-e^{-u}\}_0^t = 1 - e^{-t}.$$

26. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$,

$$f(t) = \frac{1}{2}\int_0^t \sin(t-u)\sin 2u\,du = \frac{1}{4}\int_0^t [\cos(t-3u) - \cos(t+u)]\,du$$
$$= \frac{1}{4}\left\{-\frac{1}{3}\sin(t-3u) - \sin(t+u)\right\}_0^t = -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t.$$

27. Since $\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$ and

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s+\sqrt{2}} + \frac{1/2}{s-\sqrt{2}}\right\} = \frac{1}{2}(e^{-\sqrt{2}t} + e^{\sqrt{2}t}),$$

it follows that

$$f(t) = \frac{1}{2}\int_0^t (e^{-\sqrt{2}u} + e^{\sqrt{2}u})e^{-4(t-u)}\,du = \frac{1}{2}\int_0^t \left[e^{-4t+(4-\sqrt{2})u} + e^{-4t+(4+\sqrt{2})u}\right]du$$
$$= \frac{1}{2}\left\{\frac{e^{-4t+(4-\sqrt{2})u}}{4-\sqrt{2}} + \frac{e^{-4t+(4+\sqrt{2})u}}{4+\sqrt{2}}\right\}_0^t = \frac{1}{2}\left[\frac{e^{-\sqrt{2}t} - e^{-4t}}{4-\sqrt{2}} + \frac{e^{\sqrt{2}t} - e^{-4t}}{4+\sqrt{2}}\right]$$
$$= \frac{1}{2}\left[\left(\frac{4+\sqrt{2}}{14}\right)e^{-\sqrt{2}t} + \left(\frac{4-\sqrt{2}}{14}\right)e^{\sqrt{2}t} + \left(-\frac{4+\sqrt{2}}{14} - \frac{4-\sqrt{2}}{14}\right)e^{-4t}\right]$$
$$= \left(\frac{4+\sqrt{2}}{28}\right)e^{-\sqrt{2}t} + \left(\frac{4-\sqrt{2}}{28}\right)e^{\sqrt{2}t} - \frac{2}{7}e^{-4t}.$$

28. Since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1/2}{s+2}+\dfrac{1/2}{s-2}\right\} = \dfrac{1}{2}(e^{-2t}+e^{2t})$, and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-9}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-1/6}{s+3}+\dfrac{1/6}{s-3}\right\} = \dfrac{1}{6}(e^{3t}-e^{-3t})$, it follows that

$$\begin{aligned}
f(t) &= \frac{1}{12}\int_0^t [e^{-2(t-u)}+e^{2(t-u)}][e^{3u}-e^{-3u}]\,du = \frac{1}{12}\int_0^t (e^{5u-2t}+e^{u+2t}-e^{-u-2t}-e^{2t-5u})\,du \\
&= \frac{1}{12}\left\{\frac{1}{5}e^{5u-2t}+e^{u+2t}+e^{-u-2t}+\frac{1}{5}e^{2t-5u}\right\}_0^t \\
&= \frac{1}{12}\left(\frac{1}{5}e^{3t}+e^{3t}+e^{-3t}+\frac{1}{5}e^{-3t}-\frac{1}{5}e^{-2t}-e^{2t}-e^{-2t}-\frac{1}{5}e^{2t}\right) \\
&= \frac{1}{10}(e^{3t}+e^{-3t}-e^{2t}-e^{-2t}).
\end{aligned}$$

29. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + 4Y = \frac{2}{s^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As+B-2A}{s^2-2s+4} + \frac{2}{s^3(s^2-2s+4)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{As+B-2A}{s^2-2s+4}+\frac{2}{s^3(s^2-2s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{As+B-2A}{s^2-2s+4}+\frac{1/4}{s^2}+\frac{1/2}{s^3}-\frac{1/4}{s^2-2s+4}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{As+C}{(s-1)^2+3}+\frac{1/4}{s^2}+\frac{1/2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{A(s-1)+D}{(s-1)^2+3}+\frac{1/4}{s^2}+\frac{1/2}{s^3}\right\} \\
&= e^t\mathcal{L}^{-1}\left\{\frac{As+D}{s^2+3}\right\}+\frac{t}{4}+\frac{t^2}{4} = e^t\left(A\cos\sqrt{3}t+\frac{D}{\sqrt{3}}\sin\sqrt{3}t\right)+\frac{t}{4}+\frac{t^2}{4} \\
&= e^t(A\cos\sqrt{3}t+E\sin\sqrt{3}t)+\frac{t}{4}+\frac{t^2}{4}.
\end{aligned}$$

30. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + Y = \frac{2}{(s-1)^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As+B-2A}{s^2-2s+1} + \frac{2}{(s-1)^3(s^2-2s+1)} = \frac{As+B-2A}{(s-1)^2} + \frac{2}{(s-1)^5}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{As+B-2A}{(s-1)^2}+\frac{2}{(s-1)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{A(s-1)+B-A}{(s-1)^2}+\frac{2}{(s-1)^5}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s-1}+\frac{B-A}{(s-1)^2}+\frac{2}{(s-1)^5}\right\} = Ae^t+(B-A)te^t+\frac{2}{4!}t^4e^t \\
&= Ae^t+Cte^t+\frac{t^4}{12}e^t.
\end{aligned}$$

31. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 + 1} + \frac{F(s)}{s^2 + 1}.$$

The inverse transform of this function is the solution of the differential equation

$$y(t) = \mathcal{L}^{-1}\left\{\frac{As + B}{s^2 + 1} + \frac{F(s)}{s^2 + 1}\right\} = A\cos t + B\sin t + \int_0^t f(u)\sin(t - u)\,du.$$

32. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 2[sY - A] + 5Y = \frac{1}{(s + 1)^2 + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B + 2A}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{As + B + 2A}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/3}{s^2 + 2s + 2} + \frac{As + B + 2A - 1/3}{s^2 + 2s + 5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/3}{(s + 1)^2 + 1} + \frac{A(s + 1) + B + A - 1/3}{(s + 1)^2 + 4}\right\} = \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{As + C}{s^2 + 4}\right\} \\ &= \frac{1}{3}e^{-t}\sin t + e^{-t}\left(A\cos 2t + \frac{C}{2}\sin 2t\right) = \frac{1}{3}e^{-t}\sin t + e^{-t}(A\cos 2t + D\sin 2t). \end{aligned}$$

33. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 4[sY - A] + Y = \frac{1}{s^2} + \frac{2}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B + 4A}{s^2 + 4s + 1} + \frac{2s + 1}{s^2(s^2 + 4s + 1)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{As + B + 4A}{s^2 + 4s + 1} + \frac{2s + 1}{s^2(s^2 + 4s + 1)}\right\} = \mathcal{L}^{-1}\left\{\frac{As + B + 4A}{s^2 + 4s + 1} - \frac{2}{s} + \frac{1}{s^2} + \frac{2s + 7}{s^2 + 4s + 1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{Cs + D}{s^2 + 4s + 1} - \frac{2}{s} + \frac{1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{C(s + 2) + D - 2C}{(s + 2)^2 - 3}\right\} - 2 + t \\ &= t - 2 + e^{-2t}\mathcal{L}^{-1}\left\{\frac{Cs + E}{s^2 - 3}\right\} = t - 2 + e^{-2t}\mathcal{L}^{-1}\left\{\frac{F}{s + \sqrt{3}} + \frac{G}{s - \sqrt{3}}\right\} \\ &= t - 2 + e^{-2t}(Fe^{-\sqrt{3}t} + Ge^{\sqrt{3}t}) = t - 2 + Fe^{-(2+\sqrt{3})t} + Ge^{(-2+\sqrt{3})t}. \end{aligned}$$

34. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 4Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 - 4} + \frac{F(s)}{s^2 - 4}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{As + B}{s^2 - 4} + \frac{F(s)}{s^2 - 4}\right\} = \mathcal{L}^{-1}\left\{\frac{As + B}{(s+2)(s-2)} + \frac{F(s)}{(s+2)(s-2)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{C}{s+2} + \frac{D}{s-2} + \left(\frac{-1/4}{s+2} + \frac{1/4}{s-2}\right)F(s)\right\} \\ &= Ce^{-2t} + De^{2t} + \frac{1}{4}\int_0^t [e^{2(t-u)} - e^{-2(t-u)}]f(u)\,du. \end{aligned}$$

35. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution of $y'' + 9y = te^{ti}$ satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 9Y = \frac{1}{(s-i)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 + 9} + \frac{1}{(s-i)^2(s^2+9)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{As + B}{s^2 + 9} + \frac{1}{(s-i)^2(s^2+9)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{As + B}{s^2 + 9} - \frac{i/32}{s-i} + \frac{1/8}{(s-i)^2} + \frac{-is/32 - 5/32}{s^2+9}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{Cs + D}{s^2 + 9} - \frac{i/32}{s-i} + \frac{1/8}{(s-i)^2}\right\} = C\cos 3t + \frac{D}{3}\sin 3t - \frac{i}{32}e^{ti} + \frac{t}{8}e^{ti}. \end{aligned}$$

Since C and D can be complex, if we take imaginary parts, we get

$$y(t) = E\cos 3t + F\sin 3t - \frac{1}{32}\cos t + \frac{t}{8}\sin t.$$

36. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution of $y'' - 2y' + 3y = te^{2ti}$ satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + 3Y = \frac{1}{(s-2i)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B - 2A}{s^2 - 2s + 3} + \frac{1}{(s-2i)^2(s^2 - 2s + 3)}.$$

The inverse transform of this function is the solution of the differential equation

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{As+B-2A}{s^2-2s+3}+\frac{1}{(s-2i)^2(s^2-2s+3)}\right\}\\
&\mathcal{L}^{-1}\left\{\frac{As+B-2A}{s^2-2s+3}+\frac{-62/289+44i/289}{s-2i}+\frac{-1/17+4i/17}{(s-2i)^2}+\frac{Cs+D}{s^2-2s+3}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{Es+F}{s^2-2s+3}+\frac{-62/289+44i/289}{s-2i}+\frac{-1/17+4i/17}{(s-2i)^2}\right\}\\
&= \left(-\frac{62}{289}+\frac{44i}{289}\right)e^{2ti}+\left(-\frac{1}{17}+\frac{4i}{17}\right)te^{2ti}+\mathcal{L}^{-1}\left\{\frac{E(s-1)+F+E}{(s-1)^2+2}\right\}\\
&= \left(-\frac{62}{289}+\frac{44i}{289}\right)e^{2ti}+\left(-\frac{1}{17}+\frac{4i}{17}\right)te^{2ti}+e^t\mathcal{L}^{-1}\left\{\frac{Es+G}{s^2+2}\right\}\\
&= \left(-\frac{62}{289}+\frac{44i}{289}\right)e^{2ti}+\left(-\frac{1}{17}+\frac{4i}{17}\right)te^{2ti}+e^t\left(E\cos\sqrt{2}t+\frac{G}{\sqrt{2}}\sin\sqrt{2}t\right).
\end{aligned}
$$

Constants E and G could be complex, but if we take real parts, we get

$$
y(t) = -\frac{62}{289}\cos 2t-\frac{44}{289}\sin 2t-\frac{t}{17}\cos 2t-\frac{4t}{17}\sin 2t+e^t(H\cos\sqrt{2}t+J\sin\sqrt{2}t).
$$

37. Assuming that the solution satisfies the conditions of Corollary 6.6.2, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$
[s^3Y-s^2(1)+2]-3[s^2Y-s(1)]+3[sY-1]-Y=\frac{2}{(s-1)^3}.
$$

We solve this for the transform $Y(s)$,

$$
Y(s)=\frac{s^2-3s+1}{s^3-3s^2+3s-1}+\frac{2}{(s-1)^3(s^3-3s^2+3s-1)}=\frac{s^2-3s+1}{(s-1)^3}+\frac{2}{(s-1)^6}.
$$

The inverse transform of this function is the solution of the initial-value problem

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s^2-3s+1}{(s-1)^3}+\frac{2}{(s-1)^6}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s-1}-\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}+\frac{2}{(s-1)^6}\right\}\\
&= e^t\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s^2}-\frac{1}{s^3}+\frac{2}{s^6}\right\}=e^t\left(1-t-\frac{t^2}{2}+\frac{t^5}{60}\right).
\end{aligned}
$$

38. We set $y(0)=A$, $y'(0)=B$, and $y''(0)=C$. Assuming that the solution satisfies the conditions of Corollary 6.6.2, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$
[s^3Y-As^2-Bs-C]-3[s^2Y-As-B]+3[sY-A]-Y=\frac{2}{(s-1)^3}.
$$

We solve this for the transform $Y(s)$,

$$
\begin{aligned}
Y(s) &= \frac{As^2+(B-3A)s+(C-3B+3A)}{s^3-3s^2+3s-1}+\frac{2}{(s-1)^3(s^3-3s^2+3s-1)}\\
&= \frac{As^2+(B-3A)s+(C-3B+3A)}{(s-1)^3}+\frac{2}{(s-1)^6}.
\end{aligned}
$$

The inverse transform of this function is the solution of the differential equation

$$
y(t)=\mathcal{L}^{-1}\left\{\frac{As^2+(B-3A)s+(C-3B+3A)}{(s-1)^3}+\frac{2}{(s-1)^6}\right\}
$$

$$= \mathcal{L}^{-1}\left\{\frac{D}{s-1} + \frac{E}{(s-1)^2} + \frac{F}{(s-1)^3} + \frac{2}{(s-1)^6}\right\}$$
$$= e^t\mathcal{L}^{-1}\left\{\frac{D}{s} + \frac{E}{s^2} + \frac{F}{s^3} + \frac{2}{s^6}\right\} = e^t\left(D + Et + \frac{Ft^2}{2} + \frac{t^5}{60}\right)$$
$$= e^t\left(D + Et + Gt^2 + \frac{t^5}{60}\right).$$

39. The initial-value problem is

$$\frac{1}{5}\frac{d^2x}{dt^2} + 10x = 0 \quad \Longrightarrow \quad x'' + 50x = 0, \qquad x(0) = -0.03, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X + 0.03s] + 50X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = -\frac{0.03s}{s^2+50}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{-\frac{0.03s}{s^2+50}\right\} = -0.03\cos 5\sqrt{2}t \text{ m}.$$

40. The initial-value problem is

$$2\frac{d^2x}{dt^2} + 16x = 0, \qquad x(0) = 0.1, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X - 0.1s] + 16X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{0.2s}{2s^2+16} = \frac{0.1s}{s^2+8}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{0.1s}{s^2+8}\right\} = 0.1\cos 2\sqrt{2}t \text{ m}.$$

41. The initial-value problem is

$$\frac{1}{5}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 10x = 0 \quad \Longrightarrow \quad 2x'' + 15x' + 100x = 0, \qquad x(0) = -0.03, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X + 0.03s] + 15[sX + 0.03] + 100X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = -\frac{0.06s + 0.45}{2s^2 + 15s + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{-\frac{0.06s+0.45}{2s^2+15s+100}\right\} = -\mathcal{L}^{-1}\left\{\frac{0.03s+0.225}{s^2+15s/2+50}\right\}$$
$$= -\mathcal{L}^{-1}\left\{\frac{0.03(s+15/4)+0.1125}{(s+15/4)^2+575/16}\right\} = -e^{-15t/4}\mathcal{L}^{-1}\left\{\frac{0.03s+0.1125}{s^2+575/16}\right\}$$
$$= -e^{-15t/4}\left[0.03\cos\frac{5\sqrt{23}t}{4} + 0.1125\left(\frac{4}{5\sqrt{23}}\right)\sin\frac{5\sqrt{23}t}{4}\right]$$
$$= -0.03e^{-15t/4}\left(\cos\frac{5\sqrt{23}t}{4} + \frac{3}{\sqrt{23}}\sin\frac{5\sqrt{23}t}{4}\right) \text{ m.}$$

42. The initial-value problem is

$$\frac{1}{5}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 10x = 4\sin 10t \quad \Longrightarrow \quad 2x'' + 15x' + 100x = 40\sin 10t, \qquad x(0) = 0, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X] + 15[sX] + 100X = \frac{400}{s^2+100}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{400}{(s^2+100)(2s^2+15s+100)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{400}{(s^2+100)(2s^2+15s+100)}\right\} = \mathcal{L}^{-1}\left\{\frac{-12s/65-80/65}{s^2+100} + \frac{24s/65+340/65}{2s^2+15s+100}\right\}$$
$$= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{1}{65}\mathcal{L}^{-1}\left\{\frac{12s+170}{s^2+15s/2+50}\right\}$$
$$= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{1}{65}\mathcal{L}^{-1}\left\{\frac{12(s+15/4)+125}{(s+15/4)^2+575/16}\right\}$$
$$= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{e^{-15t/4}}{65}\mathcal{L}^{-1}\left\{\frac{12s+125}{s^2+575/16}\right\}$$
$$= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{e^{-15t/4}}{65}\left(12\cos\frac{5\sqrt{23}t}{4} + \frac{100}{\sqrt{23}}\sin\frac{5\sqrt{23}t}{4}\right) \text{ m.}$$

43. The initial-value problem is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{20}\frac{dx}{dt} + 5x = 0 \quad \Longrightarrow \quad 2x'' + x' + 100x = 0, \qquad x(0) = -\frac{1}{20}, \quad x'(0) = 2.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2\left[s^2X + \frac{s}{20} - 2\right] + \left[sX + \frac{1}{20}\right] + 100X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{-s/10+79/20}{2s^2+s+100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{-s/10+79/20}{2s^2+s+100}\right\} = \frac{1}{40}\mathcal{L}^{-1}\left\{\frac{-2s+79}{s^2+s/2+50}\right\}\\
&= \frac{1}{40}\mathcal{L}^{-1}\left\{\frac{-2(s+1/4)+159/2}{(s+1/4)^2+799/16}\right\} = \frac{e^{-t/4}}{40}\mathcal{L}^{-1}\left\{\frac{-2s+159/2}{s^2+799/16}\right\}\\
&= \frac{e^{-t/4}}{40}\left[-2\cos\frac{\sqrt{799}t}{4} + \frac{159}{2}\left(\frac{4}{\sqrt{799}}\right)\sin\frac{\sqrt{799}t}{4}\right]\\
&= \frac{e^{-t/4}}{20}\left(\frac{159}{\sqrt{799}}\sin\frac{\sqrt{799}t}{4} - \cos\frac{\sqrt{799}t}{4}\right)\ \text{m}.
\end{aligned}$$

44. The initial-value problem is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 4000x = 3\cos 200t \quad\Longrightarrow\quad x'' + 40\,000x = 30\cos 200t, \qquad x(0)=0, \quad x'(0)=10.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X - 10] + 40\,000X = \frac{30s}{s^2+40\,000}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{10}{s^2+40\,000} + \frac{30s}{(s^2+40\,000)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{10}{s^2+40\,000} + \frac{30s}{(s^2+40\,000)^2}\right\} = \frac{10}{200}\sin 200t + \frac{30t}{400}\sin 200t\\
&= \frac{1}{20}\sin 200t + \frac{3t}{40}\sin 200t\ \text{m}.
\end{aligned}$$

45. The initial-value problem is

$$\frac{d^2x}{dt^2} + 64x = 2\sin 8t, \qquad x(0)=0, \quad x'(0)=0.$$

Assuming that the solution satisfies the conditions of Corollary 6.6.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X] + 64X = \frac{16}{s^2+64}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{16}{(s^2+64)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{16}{(s^2+64)^2}\right\} = \frac{16}{2(8)^3}(\sin 8t - 8t\cos 8t) = \frac{1}{64}(\sin 8t - 8t\cos 8t)\ \text{m}.$$

46. (a) Since $\sin^2 at = (1-\cos 2at)/2$,

$$\mathcal{L}\{\sin^2 at\} = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4a^2}\right) = \frac{2a^2}{s(s^2+4a^2)}.$$

(b) The derivative of $f(t) = \sin^2 at$ is $f'(t) = 2a \sin at \cos at = a \sin 2at$, and we know the Laplace transform of $\sin 2at$. We therefore take transforms of both sides of the equation and use equation 6.17 on the left,

$$sF(s) - 0 = \mathcal{L}\{a \sin 2at\} = \frac{2a^2}{s^2 + 4a^2} \quad \Longrightarrow \quad F(s) = \frac{2a^2}{s(s^2 + 4a^2)}.$$

47. The first and second derivatives of $f(t) = t \cos at$ are

$$f'(t) = \cos at - at \sin at, \qquad f''(t) = -a^2 t \cos at - 2a \sin at.$$

Since the second of these involves $\sin at$, whose transform we know, and the original function, we take Laplace transforms of this equation and use equation 6.18,

$$s^2 F(s) - 1 = -a^2 F(s) - \frac{2a^2}{s^2 + a^2} \quad \Longrightarrow \quad (s^2 + a^2)F(s) = 1 - \frac{2a^2}{s^2 + a^2} = \frac{s^2 - a^2}{s^2 + a^2}.$$

Thus, $\mathcal{L}\{t \cos at\} = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$.

48. When $n = 0$, $t^0 = 1$, and

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left\{ \frac{e^{-st}}{-s} \right\}_0^\infty = \frac{1}{s}, \quad \text{provided } s > 0.$$

The formula $\mathcal{L}\{t^n\} = n!/s^{n+1}$ is therefore valid for $n = 0$. Assume that it is valid for some integer k; that is, assume that

$$\mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}}.$$

If we take Laplace transforms of

$$\frac{d}{dt}(t^{k+1}) = (k + 1)t^k,$$

and use equation 6.17, we get

$$s\mathcal{L}\{t^{k+1}\} = (k + 1)\mathcal{L}\{t^k\} = (k + 1)\left(\frac{k!}{s^{k+1}}\right) \quad \Longrightarrow \quad \mathcal{L}\{t^{k+1}\} = \frac{(k + 1)!}{s^{k+2}}.$$

Since this is the formula for $k + 1$, we have proved that $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ for all $n \geq 0$.

49. (a) If we take Laplace transforms of $\dfrac{d}{dt}\sqrt{t} = \dfrac{1}{2\sqrt{t}}$, and use equation 6.17, we obtain

$$s\mathcal{L}\{\sqrt{t}\} = \mathcal{L}\left\{\frac{1}{2\sqrt{t}}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{s}} \quad \Longrightarrow \mathcal{L}\{\sqrt{t}\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

(b) If we take Laplace transforms of $\dfrac{d}{dt}t^{3/2} = \dfrac{3}{2}\sqrt{t}$, and use equation 6.17, we obtain

$$s\mathcal{L}\{t^{3/2}\} = \mathcal{L}\left\{\frac{3}{2}\sqrt{t}\right\} = \frac{3}{2}\frac{\sqrt{\pi}}{2s^{3/2}}.$$

Thus, $\mathcal{L}\{t^{3/2}\} = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$. If we now repeat this with $t^{5/2}$,

$$s\mathcal{L}\{t^{5/2}\} = \mathcal{L}\left\{\frac{5}{2}t^{3/2}\right\} = \frac{5}{2}\frac{3\sqrt{\pi}}{4s^{5/2}}.$$

Hence, $\mathcal{L}\{t^{5/2}\} = \dfrac{3 \cdot 5\sqrt{\pi}}{8s^{7/2}}$. The pattern emerging is

$$\mathcal{L}\{t^{(2n+1)/2}\} = \frac{1\cdot 3\cdot 5\cdots(2n+1)\sqrt{\pi}}{2^{n+1}s^{(2n+3)/2}} = \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!s^{(2n+3)/2}}.$$

This could be proved by mathematical induction.

50. (a) The proof is fashioned after that in Theorem 6.6. Suppose that t_j, $j = 1,\ldots,n$ denote the discontinuities of f' in $0 \le t \le T$. To ease notation, we shall assume that $t_0 < t_1$. Then,

$$\int_0^T e^{-st} f'(t)\,dt = \int_0^{t_0} e^{-st} f'(t)\,dt + \int_{t_0}^{t_1} e^{-st} f'(t)\,dt + \sum_{j=1}^{n}\int_{t_j}^{t_{j+1}} e^{-st} f'(t)\,dt.$$

Since f' is continuous on each subinterval, we may integrate by parts on these subintervals,

$$\int_0^T e^{-st} f'(t)\,dt = \left[\left\{e^{-st}f(t)\right\}_0^{t_0} + s\int_0^{t_0} e^{-st}f(t)\,dt\right] + \left[\left\{e^{-st}f(t)\right\}_{t_0}^{t_1} + s\int_{t_0}^{t_1} e^{-st}f(t)\,dt\right]$$
$$+\sum_{j=1}^{n}\left[\left\{e^{-st}f(t)\right\}_{t_j}^{t_{j+1}} + s\int_{t_j}^{t_{j+1}} e^{-st}f(t)\,dt\right].$$

Because f is continuous at t_j for $j > 0$, it follows that $f(t_j+) = f(t_j-)$, $j = 1,\ldots,n$, and therefore

$$\int_0^T e^{-st} f'(t)\,dt = \left[e^{-st_0}f(t_0-) - f(0) + s\int_0^{t_0} e^{-st}f(t)\,dt\right] + \left[-e^{-st_0}f(t_0+) + s\int_{t_0}^{t_1} e^{-st}f(t)\,dt\right]$$
$$+\sum_{j=1}^{n} s\int_{t_j}^{t_{j+1}} e^{-st}f(t)\,dt$$
$$= -f(0) - e^{-st_0}[f(t_0+) - f(t_0-)] + e^{-sT}f(T) + s\int_0^T e^{-st}f(t)\,dt.$$

Thus,

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{T\to\infty}\int_0^T e^{-st}f'(t)\,dt$$
$$= \lim_{T\to\infty}\left[-f(0) - e^{-st_0}[f(t_0+) - f(t_0-)] + e^{-sT}f(T) + s\int_0^T e^{-st}f(t)\,dt\right]$$
$$= sF(s) - f(0) - e^{-st_0}[f(t_0+) - f(t_0-)] + \lim_{T\to\infty} e^{-sT}f(T),$$

provided the limit on the right exists. Since f is $O(e^{\alpha t})$, there exists M and $\overline{T}$ such that for $t > \overline{T}$, $|f(t)| < Me^{\alpha t}$. Thus, for $T > \overline{T}$,

$$e^{-sT}|f(T)| < e^{-sT}Me^{\alpha T} = Me^{(\alpha-s)T}$$

which approaches 0 as $T \to \infty$ (provided $s > \alpha$). Consequently,

$$\mathcal{L}\{f'\} = sF(s) - f(0) - e^{-st_0}[f(t_0+) - f(t_0-)].$$

(b) When $t_0 = 0$, the result becomes

$$\mathcal{L}\{f'\} = sF(s) - f(t_0+).$$

(c) When $f(t)$ has discontinuities at t_n,

$$\mathcal{L}\{f(t)\} = sF(s) - \sum_{n=1}^{\infty} e^{-st_n}[f(t_n+) - f(t_n-)].$$

51. (a) Instead of using the definition for the convolution of t^n and t^m, we note that

$$\mathcal{L}\{t^n * t^m\} = \mathcal{L}\{t^n\}\,\mathcal{L}\{t^m\} = \left(\frac{n!}{s^{n+1}}\right)\left(\frac{m!}{s^{m+1}}\right) = \frac{n!\,m!}{s^{n+m+2}}.$$

When we take inverse transforms, we get

$$t^n * t^m = \frac{n!\,m!}{(n+m+1)!}t^{m+n+1}.$$

(b) When m and n are not positive integers, we use the result of Exercise 44 in Section 6.1,

$$\mathcal{L}\{t^n * t^m\} = \mathcal{L}\{t^n\}\,\mathcal{L}\{t^m\} = \left[\frac{\Gamma(n+1)}{s^{n+1}}\right]\left[\frac{\Gamma(m+1)}{s^{m+1}}\right] = \frac{\Gamma(n+1)\Gamma(m+1)}{s^{n+m+2}}.$$

When we take inverse transforms, we get

$$t^n * t^m = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)}t^{m+n+1}.$$

52. When $f(t)$ satisfies the conditions of Theorem 6.6, the Laplace transform of $f'(t)$ exists for $s > \alpha$, and it is given by $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$. According to Theorem 53 of Section 6.1, the limit of $\mathcal{L}\{f'(t)\}$ must be zero as $s \to \infty$. This yields the required result.

53. If we take limits as $s \to 0$ on terms in equation 6.17,

$$\lim_{s\to 0}\mathcal{L}\{f'(t)\} = \lim_{s\to 0}\,[s\,F(s)] - f(0).$$

Thus,

$$\lim_{s\to 0}\,[s\,F(s)] = \lim_{s\to 0}\left[\int_0^\infty e^{-st}f'(t)\,dt\right] + f(0) = \lim_{s\to 0}\left[\lim_{T\to\infty}\int_0^T e^{-st}f'(t)\,dt\right] + f(0).$$

If we interchange the order of limits, we get

$$\begin{aligned}\lim_{s\to 0}\,[s\,F(s)] &= \lim_{T\to\infty}\left[\lim_{s\to 0}\int_0^T e^{-st}f'(t)\,dt\right] + f(0) = \lim_{T\to\infty}\int_0^T f'(t)\,dt + f(0)\\ &= \lim_{T\to\infty}\,[f(T)] - f(0) + f(0) = \lim_{t\to\infty} f(t).\end{aligned}$$

54. The function $f(t) = \sin(e^{t^2})$ is continous and of exponential order zero. Since its derivative $f'(t) = 2te^{t^2}\cos(e^{t^2})$ is continuous, Theorem 6.17 guarantees that the Laplace transform of $f'(t)$ exists.

EXERCISES 6.4

1. (a) On the interval $0 < t < 1$, the differential equation $dy/dt + 3y = t$ has integraing factor e^{3t}, so that

$$\frac{d}{dt}(ye^{3t}) = te^{3t} \implies ye^{3t} = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + C \implies y(t) = \frac{t}{3} - \frac{1}{9} + Ce^{-3t}.$$

The initial condition requires $1 = y(0) = -1/9 + C$. Hence, for $0 < t < 1$, $y(t) = \frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t}$. For $t > 1$, a general solution of the differential equation is $y(t) = 1/3 + De^{-3t}$. For the solution to be continuous at $t = 1$, we must have

$$\lim_{t\to 1^-} y(t) = \lim_{t\to 1^+} y(t) \implies \frac{1}{3} - \frac{1}{9} + \frac{10}{9}e^{-3} = \frac{1}{3} + De^{-3} \implies D = \frac{1}{9}(10 - e^3).$$

Thus,

$$y(t) = \begin{cases} \dfrac{t}{3} - \dfrac{1}{9} + \dfrac{10}{9}e^{-3t}, & 0 < t \le 1 \\ \dfrac{1}{3} + \dfrac{1}{9}(10 - e^3)e^{-3t}, & t > 1. \end{cases}$$

(b) When we write $f(t) = t[h(t) - h(t-1)] + h(t-1) = t + (1-t)h(t-1)$, and take Laplace transforms of the differential equation,

$$sY - 1 + 3Y = \frac{1}{s^2} + e^{-s}\mathcal{L}\{1 - (t+1)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

Thus,

$$Y(s) = \frac{1}{s+3}\left(1 + \frac{1}{s^2} - \frac{e^{-s}}{s^2}\right).$$

Consequently,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+3} + \frac{1}{s^2(s+3)} - \frac{e^{-s}}{s^2(s+3)}\right\} \\ &= e^{-3t} + \mathcal{L}^{-1}\left\{\left(\frac{1/3}{s^2} - \frac{1/9}{s} + \frac{1/9}{s+3}\right) - e^{-s}\left(\frac{1/3}{s^2} - \frac{1/9}{s} + \frac{1/9}{s+3}\right)\right\} \\ &= e^{-3t} + \frac{t}{3} - \frac{1}{9} + \frac{1}{9}e^{-3t} - \left[\frac{1}{3}(t-1) - \frac{1}{9} + \frac{1}{9}e^{-3(t-1)}\right]h(t-1) \\ &= \frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t} + \left[-\frac{t}{3} + \frac{4}{9} - \frac{1}{9}e^{-3(t-1)}\right]h(t-1). \end{aligned}$$

2. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 9Y = \mathcal{L}\{h(t-4)\} = \frac{e^{-4s}}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+2}{s^2+9} + \frac{e^{-4s}}{s(s^2+9)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + \frac{e^{-4s}}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + \left(\frac{1/9}{s} + \frac{-s/9}{s^2+9}\right)e^{-4s}\right\} \\ &= \cos 3t + \frac{2}{3}\sin 3t + \frac{1}{9}[1 - \cos 3(t-4)]h(t-4). \end{aligned}$$

3. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 9Y = \mathcal{L}\{2[h(t) - h(t-4)]\} = 2\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+2}{s^2+9} + \frac{2(1-e^{-4s})}{s(s^2+9)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + \frac{2(1-e^{-4s})}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + 2\left(\frac{1/9}{s} + \frac{-s/9}{s^2+9}\right)(1-e^{-4s})\right\} \\ &= \cos 3t + \frac{2}{3}\sin 3t + \frac{2}{9}(1-\cos 3t) - \frac{2}{9}[1-\cos 3(t-4)]h(t-4). \\ &= \frac{2}{9} + \frac{7}{9}\cos 3t + \frac{2}{3}\sin 3t - \frac{2}{9}[1-\cos 3(t-4)]h(t-4). \end{aligned}$$

4. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y + 1] + 4[sY] + 4Y &= \mathcal{L}\{t[h(t) - h(t-1)] + h(t-1)\} = \mathcal{L}\{t + (1-t)h(t-1)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{-t\} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{-1}{s^2+4s+4} + \frac{1-e^{-s}}{s^2(s^2+4s+4)} = \frac{-1}{(s+2)^2} + \frac{1-e^{-s}}{s^2(s+2)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)^2} + \frac{1-e^{-s}}{s^2(s+2)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)^2} + \left[\frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2}\right](1-e^{-s})\right\} \\ &= -te^{-2t} + \frac{1}{4}(-1 + t + e^{-2t} + te^{-2t}) - \frac{1}{4}\left[-1 + (t-1) + e^{-2(t-1)} + (t-1)e^{-2(t-1)}\right]h(t-1) \\ &= \frac{1}{4}(-1 + t + e^{-2t} - 3te^{-2t}) + \frac{1}{4}[2 - t - te^{2(1-t)}]h(t-1). \end{aligned}$$

5. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y - s(-1)] + 4[sY + 1] + 4Y &= \mathcal{L}\{(2-t)[h(t) - h(t-2)] + (t-2)h(t-2)\} \\ &= \mathcal{L}\{2 - t + 2(t-2)h(t-2)\} = \frac{2}{s} - \frac{1}{s^2} + 2e^{-2s}\mathcal{L}\{t\} \\ &= \frac{2}{s} - \frac{1}{s^2} + \frac{2e^{-2s}}{s^2}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = -\frac{s+4}{s^2+4s+4} + \frac{2}{s(s^2+4s+4)} - \frac{1-2e^{-2s}}{s^2(s^2+4s+4)} = -\frac{s+4}{(s+2)^2} + \frac{2}{s(s+2)^2} - \frac{1-2e^{-2s}}{s^2(s+2)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{-\frac{s+4}{(s+2)^2}+\frac{2}{s(s+2)^2}-\frac{1-2e^{-2s}}{s^2(s+2)^2}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1/2}{s}-\frac{3/2}{s+2}-\frac{3}{(s+2)^2}-\left[\frac{-1/4}{s}+\frac{1/4}{s^2}+\frac{1/4}{s+2}+\frac{1/4}{(s+2)^2}\right](1-2e^{-2s})\right\}\\
&= \frac{1}{2}-\frac{3}{2}e^{-2t}-3te^{-2t}+\frac{1}{4}(1-t-e^{-2t}-te^{-2t})\\
&\quad +\frac{1}{2}[-1+(t-2)+e^{-2(t-2)}+(t-2)e^{-2(t-2)}]h(t-2)\\
&= \frac{3}{4}-\frac{t}{4}-\frac{7}{4}e^{-2t}-\frac{13t}{4}e^{-2t}+\frac{1}{2}[-3+t-e^{2(2-t)}+te^{2(2-t)}]h(t-2).
\end{aligned}$$

6. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y-s(1)-2]+4[sY-1]+3Y &= \mathcal{L}\{\sin t\,h(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} = e^{-\pi s}\mathcal{L}\{-\sin t\}\\
&= -\frac{e^{-\pi s}}{s^2+1}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+6}{s^2+4s+3}-\frac{e^{-\pi s}}{(s^2+1)(s^2+4s+3)} = \frac{s+6}{(s+1)(s+3)}-\frac{e^{-\pi s}}{(s^2+1)(s+1)(s+3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s+6}{(s+1)(s+3)}-\frac{e^{-\pi s}}{(s^2+1)(s+1)(s+3)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{5/2}{s+1}-\frac{3/2}{s+3}-\left(\frac{1/4}{s+1}-\frac{1/20}{s+3}+\frac{-s/5+1/10}{s^2+1}\right)e^{-\pi s}\right\}\\
&= \frac{5}{2}e^{-t}-\frac{3}{2}e^{-3t}+\frac{1}{20}\left[-5e^{-(t-\pi)}+e^{-3(t-\pi)}+4\cos(t-\pi)-2\sin(t-\pi)\right]h(t-\pi)\\
&= \frac{5}{2}e^{-t}-\frac{3}{2}e^{-3t}+\frac{1}{20}\left[-5e^{\pi-t}+e^{3(\pi-t)}-4\cos t+2\sin t\right]h(t-\pi).
\end{aligned}$$

7. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y-s(1)-2]+4[sY-1]+3Y &= \mathcal{L}\{\sin t[h(t)-h(t-\pi)]\} = \frac{1}{s^2+1}-e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}\\
&= \frac{1}{s^2+1}-e^{-\pi s}\mathcal{L}\{-\sin t\} = \frac{1}{s^2+1}+\frac{e^{-\pi s}}{s^2+1}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+6}{s^2+4s+3}+\frac{1+e^{-\pi s}}{(s^2+1)(s^2+4s+3)} = \frac{s+6}{(s+1)(s+3)}+\frac{1+e^{-\pi s}}{(s^2+1)(s+1)(s+3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s+6}{(s+1)(s+3)}+\frac{1+e^{-\pi s}}{(s^2+1)(s+1)(s+3)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{5/2}{s+1}-\frac{3/2}{s+3}+\left(\frac{1/4}{s+1}-\frac{1/20}{s+3}+\frac{-s/5+1/10}{s^2+1}\right)(1+e^{-\pi s})\right\}\\
&= \frac{5}{2}e^{-t}-\frac{3}{2}e^{-3t}+\frac{1}{20}\left[5e^{-t}-e^{-3t}-4\cos t+2\sin t\right]\\
&\quad +\frac{1}{20}\left[5e^{-(t-\pi)}-e^{-3(t-\pi)}-4\cos(t-\pi)+2\sin(t-\pi)\right]h(t-\pi)\\
&= \frac{11}{4}e^{-t}-\frac{31}{20}e^{-3t}-\frac{1}{5}\cos t+\frac{1}{10}\sin t+\frac{1}{20}\left[5e^{\pi-t}-e^{3(\pi-t)}+4\cos t-2\sin t\right]h(t-\pi).
\end{aligned}$$

8. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] + 5Y = \mathcal{L}\{3[h(t) - h(t-1)] - 3h(t-1)\} = \frac{3}{s} - 6\mathcal{L}\{h(t-1)\} = \frac{3}{s} - \frac{6e^{-s}}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{3(1 - 2e^{-s})}{s(s^2 + 2s + 5)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{3(1-2e^{-s})}{s(s^2+2s+5)}\right\} = 3\mathcal{L}^{-1}\left\{\left(\frac{1/5}{s} - \frac{s/5 + 2/5}{s^2+2s+5}\right)(1 - 2e^{-s})\right\} \\ &= \frac{3}{5}\mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right](1-2e^{-s})\right\} \\ &= \frac{3}{5}\left[1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right] - \frac{6}{5}\left\{1 - e^{-(t-1)}\left[\cos 2(t-1) + \frac{1}{2}\sin 2(t-1)\right]\right\}h(t-1). \end{aligned}$$

9. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y] + 2[sY] + 5Y &= \mathcal{L}\{4[h(t) - h(t-1)] - 4[h(t-1) - h(t-2)]\} = \mathcal{L}\{4 - 8h(t-1) + 4h(t-2)\} \\ &= \frac{4}{s} - \frac{8e^{-s}}{s} + \frac{4e^{-2s}}{s}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{4}{s(s^2+2s+5)}(1 - 2e^{-s} + e^{-2s}).$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+2s+5)}(1-2e^{-s}+e^{-2s})\right\} = 4\mathcal{L}^{-1}\left\{\left(\frac{1/5}{s} - \frac{s/5+2/5}{s^2+2s+5}\right)(1-2e^{-s}+e^{-2s})\right\} \\ &= \frac{4}{5}\mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right](1-2e^{-s}+e^{-2s})\right\} \\ &= \frac{4}{5}\left[1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right] - \frac{8}{5}\left\{1 - e^{-(t-1)}\left[\cos 2(t-1) + \frac{1}{2}\sin 2(t-1)\right]\right\}h(t-1) \\ &\quad + \frac{4}{5}\left\{1 - e^{-(t-2)}\left[\cos 2(t-2) + \frac{1}{2}\sin 2(t-2)\right]\right\}h(t-2) \\ &= \frac{2}{5}\left[2 - e^{-t}(2\cos 2t + \sin 2t)\right] + \frac{4}{5}\left\{-2 + e^{1-t}\left[2\cos 2(t-1) + \sin 2(t-1)\right]\right\}h(t-1) \\ &\quad + \frac{2}{5}\left\{2 - e^{2-t}\left[2\cos 2(t-2) + \sin 2(t-2)\right]\right\}h(t-2). \end{aligned}$$

10. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y - s(2)] + 16Y &= \frac{1}{1-e^{-2s}}\mathcal{L}\{t[h(t) - h(t-1)]\} = \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\}\right] \\ &= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right] = \frac{1}{(1+e^{-s})(1-e^{-s})}\left[\frac{1}{s^2}(1-e^{-s}) - \frac{e^{-s}}{s}\right]. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s}{s^2+16} + \frac{1}{s^2(s^2+16)(1+e^{-s})} - \frac{e^{-s}}{s(s^2+16)(1-e^{-2s})}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \frac{1}{s^2(s^2+16)(1+e^{-s})} - \frac{e^{-s}}{s(s^2+16)(1-e^{-2s})}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \left(\frac{1/16}{s^2} - \frac{1/16}{s^2+16}\right)\sum_{n=0}^{\infty}(-1)^n e^{-ns} - \left(\frac{1/16}{s} - \frac{s/16}{s^2+16}\right)e^{-s}\sum_{n=0}^{\infty}e^{-2ns}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \frac{1}{16}\left(\frac{1}{s^2} - \frac{1}{s^2+16}\right)\sum_{n=0}^{\infty}(-1)^n e^{-ns} - \frac{1}{16}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)\sum_{n=0}^{\infty}e^{-(2n+1)s}\right\}\\
&= 2\cos 4t + \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n\left[(t-n) - \frac{1}{4}\sin 4(t-n)\right]h(t-n)\\
&\qquad - \frac{1}{16}\sum_{n=0}^{\infty}[1-\cos 4(t-2n-1)]\,h(t-2n-1)\\
&= 2\cos 4t + \frac{1}{64}\sum_{n=0}^{\infty}(-1)^n[4(t-n) - \sin 4(t-n)]\,h(t-n)\\
&\qquad - \frac{1}{16}\sum_{n=0}^{\infty}[1-\cos 4(t-2n-1)]\,h(t-2n-1).
\end{aligned}$$

11. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y - s(2)] + 16Y &= \frac{1}{1-e^{-2s}}\mathcal{L}\{t[h(t)-h(t-1)] + (2-t)[h(t-1)-h(t-2)]\}\\
&= \frac{1}{1-e^{-2s}}\mathcal{L}\{t + (2-2t)h(t-1) + (t-2)h(t-2)\}\\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} + e^{-s}\mathcal{L}\{2-2(t+1)\} + e^{-2s}\mathcal{L}\{t\}\right]\\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right] = \frac{(1-e^{-s})^2}{s^2(1-e^{-s})(1+e^{-s})} = \frac{1-e^{-s}}{s^2(1+e^{-s})}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s}{s^2+16} + \frac{1-e^{-s}}{s^2(s^2+16)(1+e^{-s})}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \frac{1-e^{-s}}{s^2(s^2+16)(1+e^{-s})}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \left(\frac{1/16}{s^2} - \frac{1/16}{s^2+16}\right)(1-e^{-s})\sum_{n=0}^{\infty}(-1)^n e^{-ns}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \frac{1}{16}\left(\frac{1}{s^2} - \frac{1}{s^2+16}\right)\left[\sum_{n=0}^{\infty}(-1)^n e^{-ns} + \sum_{n=0}^{\infty}(-1)^{n+1}e^{-(n+1)s}\right]\right\}\\
&= 2\cos 4t + \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n\left[(t-n) - \frac{1}{4}\sin 4(t-n)\right]h(t-n)\\
&\qquad + \frac{1}{16}\sum_{n=0}^{\infty}(-1)^{n+1}\left[(t-n-1) - \frac{1}{4}\sin 4(t-n-1)\right]h(t-n-1)
\end{aligned}$$

$$= 2\cos 4t + \frac{1}{64}\sum_{n=0}^{\infty}(-1)^n\left[4(t-n) - \sin 4(t-n)\right]h(t-n)$$
$$+ \frac{1}{64}\sum_{n=0}^{\infty}(-1)^{n+1}\left[4(t-n-1) - \sin 4(t-n-1)\right]h(t-n-1).$$

12. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$s^2Y + sY = \frac{1}{1-e^{-2s}}\mathcal{L}\{[h(t)-h(t-1)] - [h(t-1)-h(t-2)]\}$$
$$= \frac{1}{1-e^{-2s}}\left(\frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}\right) = \frac{(1-e^{-s})^2}{s(1-e^{-s})(1+e^{-s})} = \frac{1-e^{-s}}{s(1+e^{-s})}.$$

Thus,

$$Y(s) = \frac{1-e^{-s}}{s^2(s+1)(1+e^{-s})} = \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)(1-e^{-s})\sum_{n=0}^{\infty}(-e^{-s})^n$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)\left[e^{-ns} - e^{(n+1)s}\right].$$

Inverse transforms now give

$$y(t) = \sum_{n=0}^{\infty}(-1)^n\left\{\left[(t-n-1) + e^{-(t-n)}\right]h(t-n) - \left[(t-n-2) + e^{-(t-n-1)}\right]h(t-n-1)\right\}.$$

13. The initial-value problem for the number of grams $g(t)$ of salt in the tank as a function of time t in seconds is

$$\frac{dg}{dt} + \frac{g}{2\times 10^5} = \frac{1}{10}, \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 + \frac{G}{2\times 10^5} = \frac{1}{10s}.$$

We solve this for $G(s)$,

$$G(s) = \frac{5000}{s + \frac{1}{2\times 10^5}} + \frac{1}{10s\left(s + \frac{1}{2\times 10^5}\right)}$$
$$= \frac{5000}{s + \frac{1}{2\times 10^5}} + \frac{1}{10}\left(\frac{2\times 10^5}{s} - \frac{2\times 10^5}{s + \frac{1}{2\times 10^5}}\right) = \frac{20\,000}{s} - \frac{15\,000}{s + \frac{1}{200\,000}}.$$

Inverse transforms now give

$$g(t) = 20\,000 - 15\,000e^{-t/200\,000}.$$

14. Laplace transforms cannot handle the quotient $S(t)/(10^6 + 5t)$.

15. The initial-value problem for the number of grams $g(t)$ of salt in the tank as a function of time t in seconds is

$$\frac{dg}{dt} + \frac{g}{2\times 10^5} = \frac{1}{10}[h(t) - h(t-600)] + \frac{1}{20}h(t-600), \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 + \frac{G}{2\times 10^5} = \frac{1}{10}\left(\frac{1}{s} - \frac{e^{-600s}}{s}\right) + \frac{e^{-600s}}{20s}.$$

We solve this for $G(s)$,

$$G(s) = \frac{5000}{s + \frac{1}{2\times10^5}} + \frac{\frac{1}{20}\left(\frac{2}{s} - \frac{e^{-600s}}{s}\right)}{s + \frac{1}{2\times10^5}}$$
$$= \frac{5000}{s + \frac{1}{2\times10^5}} + \frac{1}{20}(2 - e^{-600s})\left(\frac{2\times10^5}{s} - \frac{2\times10^5}{s + \frac{1}{2\times10^5}}\right)$$
$$= \frac{20\,000}{s} - \frac{15\,000}{s + \frac{1}{200\,000}} - 10\,000e^{-600s}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{200\,000}}\right).$$

Inverse transforms now give

$$g(t) = 20\,000 - 15\,000e^{-t/200\,000} - 10\,000\left[1 - e^{-(t-600)/200\,000}\right]h(t-600).$$

16. The initial-value problem for the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = \frac{1000t}{T}[1 - h(t-T)], \quad g(0) = 0.$$

When we take Laplace transforms,

$$sG = \frac{1000}{Ts^2} - \frac{1000}{T}e^{-Ts}\mathcal{L}\{t+T\} - \frac{G}{100} = \frac{1000}{Ts^2} - \frac{1000}{T}e^{-Ts}\left(\frac{1}{s^2} + \frac{T}{s}\right) - \frac{G}{100}.$$

Thus,

$$G(s) = \frac{1000}{Ts^2(s+1/100)} - \frac{1000}{T(s+1/100)}\left(\frac{1}{s^2} + \frac{T}{s}\right)e^{-Ts}.$$

With partial fractions, we can invert this transform,

$$g(t) = \frac{1000}{T}\mathcal{L}^{-1}\left\{-\frac{10^4}{s} + \frac{100}{s^2} + \frac{10^4}{s+1/100}\right\} - \frac{1000}{T}\mathcal{L}^{-1}\left\{\left(-\frac{10^4}{s} + \frac{100}{s^2} + \frac{10^4}{s+1/100}\right)e^{-Ts}\right\}$$
$$- 1000\mathcal{L}^{-1}\left\{\left(\frac{100}{s} - \frac{100}{s+1/100}\right)e^{-Ts}\right\}$$
$$= \frac{10^5}{T}[t + 100(e^{-t/100} - 1)] - \frac{10^5}{T}[(t-T) + 100(e^{-(t-T)/100} - 1)]h(t-T)$$
$$- 10^5[1 - e^{-(t-T)/100}]h(t-T).$$

17. The initial-value problem for the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = f(t) - \frac{5g}{10^6}, \quad g(0) = 5000,$$

where

$$f(t) = \begin{cases} 1/10, & 0 < t < 60 \\ 0, & 60 < t < 120, \end{cases}, \qquad f(t+120) = f(t).$$

When we take Laplace transforms,

$$sG - 5000 = \frac{1}{1 - e^{-120s}}\mathcal{L}\left\{\frac{1}{10}[h(t) - h(t-60)]\right\} - \frac{5G}{10^6}$$
$$= \frac{1}{10(1 - e^{-120s})}\left(\frac{1}{s} - \frac{e^{-60s}}{s}\right) - \frac{5G}{10^6}.$$

Solving for $G(s)$ gives

$$\begin{aligned}G(s) &= \frac{1-e^{-60s}}{10s(s+5/10^6)(1-e^{-120s})} + \frac{5000}{s+5/10^6}\\ &= \frac{1}{10s(s+5/10^6)(1+e^{-60s})} + \frac{5000}{s+5/10^6}\\ &= \frac{10^5}{5}\left(\frac{1}{s} - \frac{1}{s+5/10^6}\right)\sum_{n=0}^{\infty}(-1)^n e^{-60ns} + \frac{5000}{s+5/10^6}.\end{aligned}$$

Thus,

$$g(t) = 20\,000\sum_{n=0}^{\infty}(-1)^n[1-e^{-5(t-60n)/10^6}]h(t-60n) + 5000e^{-5t/10^6}.$$

18. The initial-value problem for the number of grams $g(t)$ of glucose in the bloodstream is

$$\frac{dg}{dt} = R[h(t)-h(t-240)] - kg, \qquad g(0) = g_0,$$

where $k > 0$ is a constant. When we take Laplace transforms,

$$sG - g_0 = \frac{R}{s}(1-e^{-240s}) - kG.$$

We now solve for $G(s)$,

$$G(s) = \frac{R}{s(s+k)}(1-e^{-240s}) + \frac{g_0}{s+k} = \frac{R}{k}\left(\frac{1}{s} - \frac{1}{s+k}\right)(1-e^{-240s}) + \frac{g_0}{s+k}.$$

Inverse transforms give

$$g(t) = \frac{R}{k}(1-e^{-kt}) - \frac{R}{k}[1-e^{-k(t-240)}]h(t-240) + g_0e^{-kt}.$$

After 6 hours, the number of grams of glucose in the bloodstream is

$$g(360) = \frac{R}{k}(1-e^{-360k}) - \frac{R}{k}(1-e^{-120k}) + g_0e^{-360k} = \frac{R}{k}(e^{-120k} - e^{-360k}) + g_0e^{-360k}.$$

19. The initial-value problem for the number of grams $g(t)$ of glucose in the bloodstream is

$$\frac{dg}{dt} = Rf(t) - kg, \qquad g(0) = g_0,$$

where $k > 0$ is a constant, and f(t) is the periodic function

$$f(t) = h(t) - h(t-60) = 1 - h(t-60), \quad 0 < t < 120, \quad f(t+120) = f(t).$$

When we take Laplace transforms,

$$sG - g_0 = \frac{R}{1-e^{-120s}}\mathcal{L}\{1-h(t-60)\} - kG = \frac{R}{s(1-e^{-120s})}(1-e^{-60s}) - kG.$$

We now solve for $G(s)$,

$$G(s) = \frac{R}{s(s+k)(1+e^{-60s})} + \frac{g_0}{s+k} = \frac{R}{k}\left(\frac{1}{s} - \frac{1}{s+k}\right)\sum_{n=0}^{\infty}(-1)^n e^{-60ns} + \frac{g_0}{s+k}.$$

Inverse transforms give

$$g(t) = \frac{R}{k}\sum_{n=0}^{\infty}(-1)^n[1-e^{-k(t-60n)}]h(t-60n) + g_0e^{-kt}.$$

20. (a) If time t is measured in months, the initial-value problem for $m(t)$ is

$$\frac{dm}{dt} = km - H[h(t) - h(t-1)], \quad m(0) = m_0.$$

If we replace $h(t)$ with 1, and take Laplace transforms,

$$sM - m_0 = kM - H\left(\frac{1}{s} - \frac{e^{-s}}{s}\right).$$

When we solve for $M(s)$,

$$M(s) = \frac{m_0}{s-k} - \frac{H(1-e^{-s})}{s(s-k)} = \frac{m_0}{s-k} + \frac{H}{k}\left(\frac{1}{s} - \frac{1}{s-k}\right)(1-e^{-s}).$$

Inverse transforms give

$$m(t) = m_0e^{kt} + \frac{H}{k}\left[(1-e^{kt}) - (1-e^{k(t-1)})h(t-1)\right] \text{ kg}.$$

(b) For $m(12) = m_0$,

$$m_0 = m_0e^{12k} + \frac{H}{k}[(1-e^{12k}) - (1-e^{11k})] \qquad \Longrightarrow \qquad H = \frac{km_0(e^{12k}-1)}{e^{12k}-e^{11k}}.$$

21. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 40x = 100[h(t) - h(t-4)], \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 400x = 1000[1 - h(t-4)],$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 400X = 1000\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right) = \frac{1000(1-e^{-4s})}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 2}{s^2+400} + \frac{1000(1-e^{-4s})}{s(s^2+400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10-2}{s^2+400} + \frac{1000(1-e^{-4s})}{s(s^2+400)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s/10-2}{s^2+400} + \frac{5}{2}\left(\frac{1}{s} - \frac{s}{s^2+400}\right)(1-e^{-4s})\right\} \\
&= \frac{1}{10}\cos 20t - \frac{1}{10}\sin 20t + \frac{5}{2}(1-\cos 20t) - \frac{5}{2}[1-\cos 20(t-4)]h(t-4) \\
&= \frac{5}{2} - \frac{12}{5}\cos 20t - \frac{1}{10}\sin 20t - \frac{5}{2}[1-\cos 20(t-4)]h(t-4) \text{ m}.
\end{aligned}$$

22. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 40x = 100h(t-4), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 2}{s^2 + 400} + \frac{1000e^{-4s}}{s(s^2 + 400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 2}{s^2 + 400} + \frac{1000e^{-4s}}{s(s^2 + 400)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s/10 - 2}{s^2 + 400} + \frac{5}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 400}\right)e^{-4s}\right\} \\ &= \frac{1}{10}\cos 20t - \frac{1}{10}\sin 20t + \frac{5}{2}[1 - \cos 20(t-4)]h(t-4) \text{ m.} \end{aligned}$$

23. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 40x = 100[h(t) - h(t-4)], \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 400x = 1000[1 - h(t-4)],$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 50\left[sX - \frac{1}{10}\right] + 400X = 1000\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 + 3}{s^2 + 50s + 400} + \frac{1000(1 - e^{-4s})}{s(s^2 + 50s + 400)} = \frac{s/10 + 3}{(s+10)(s+40)} + \frac{1000(1 - e^{-4s})}{s(s+10)(s+40)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 + 3}{(s+10)(s+40)} + \frac{1000(1 - e^{-4s})}{s(s+10)(s+40)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/15}{s+10} + \frac{1/30}{s+40} + 1000\left(\frac{1/400}{s} - \frac{1/300}{s+10} + \frac{1/1200}{s+40}\right)(1 - e^{-4s})\right\} \\ &= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \frac{5}{2} - \frac{10}{3}e^{-10t} + \frac{5}{6}e^{-40t} - \left[\frac{5}{2} - \frac{10}{3}e^{-10(t-4)} + \frac{5}{6}e^{-40(t-4)}\right]h(t-4) \\ &= \frac{5}{2} - \frac{49}{15}e^{-10t} + \frac{13}{15}e^{-40t} - \frac{5}{6}\left[3 - 4e^{10(4-t)} + e^{40(4-t)}\right]h(t-4) \text{ m.} \end{aligned}$$

24. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 40x = 100h(t-4), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 50\left[sX - \frac{1}{10}\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10+3}{s^2+50s+400} + \frac{1000e^{-4s}}{s(s^2+50s+400)} = \frac{s/10+3}{(s+10)(s+40)} + \frac{1000e^{-4s}}{s(s+10)(s+40)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10+3}{(s+10)(s+40)} + \frac{1000e^{-4s}}{s(s+10)(s+40)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1/15}{s+10} + \frac{1/30}{s+40} + 1000\left(\frac{1/400}{s} - \frac{1/300}{s+10} + \frac{1/1200}{s+40}\right) - e^{-4s}\right\} \\
&= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \left[\frac{5}{2} - \frac{10}{3}e^{-10(t-4)} + \frac{5}{6}e^{-40(t-4)}\right]h(t-4) \\
&= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \frac{5}{6}\left[3 - 4e^{10(4-t)} + e^{40(4-t)}\right]h(t-4) \text{ m.}
\end{aligned}$$

25. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 40x = 100[h(t) - h(t-4)], \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 400x = 1000[1 - h(t-4)],$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 10\left[sX - \frac{1}{10}\right] + 400X = 1000\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10-1}{s^2+10s+400} + \frac{1000(1-e^{-4s})}{s(s^2+10s+400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10-1}{s^2+10s+400} + \frac{1000(1-e^{-4s})}{s(s^2+10s+400)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{10}\left(\frac{s-10}{s^2+10s+400}\right) + 1000\left(\frac{1/400}{s} - \frac{s/400+1/40}{s^2+10s+400}\right)(1-e^{-4s})\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{10}\left[\frac{(s+5)-15}{(s+5)^2+375}\right] + \frac{5}{2}\left[\frac{1}{s} - \frac{(s+5)+5}{(s+5)^2+375}\right](1-e^{-4s})\right\} \\
&= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right) + \frac{5}{2}\left[1 - e^{-5t}\left(\cos 5\sqrt{15}t + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}t\right)\right] \\
&\qquad - \frac{5}{2}\left\{1 - e^{-5(t-4)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4) \\
&= \frac{5}{2} - \frac{12}{5}e^{-5t}\cos 5\sqrt{15}t - \frac{14\sqrt{15}}{75}e^{-5t}\sin 5\sqrt{15}t \\
&\qquad - \frac{5}{2}\left\{1 - e^{5(4-t)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4) \text{ m.}
\end{aligned}$$

26. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 40x = 100h(t-4), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 10\left[sX - \frac{1}{10}\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 1}{s^2 + 10s + 400} + \frac{1000e^{-4s}}{s(s^2 + 10s + 400)}.$$

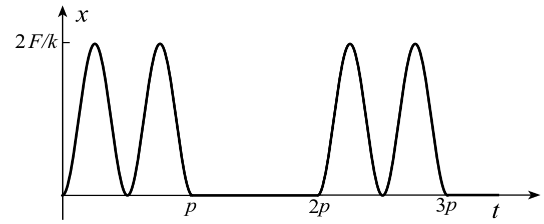
The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 1}{s^2 + 10s + 400} + \frac{1000e^{-4s}}{s(s^2 + 10s + 400)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{10}\left(\frac{s - 10}{s^2 + 10s + 400}\right) + 1000\left(\frac{1/400}{s} - \frac{s/400 + 1/40}{s^2 + 10s + 400}\right)e^{-4s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{10}\left[\frac{(s+5) - 15}{(s+5)^2 + 375}\right] + \frac{5}{2}\left[\frac{1}{s} - \frac{(s+5) + 5}{(s+5)^2 + 375}\right]e^{-4s}\right\} \\
&= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right) \\
&\qquad + \frac{5}{2}\left\{1 - e^{-5(t-4)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4) \\
&= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right) \\
&\qquad + \frac{5}{2}\left\{1 - e^{5(4-t)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4) \text{ m.}
\end{aligned}$$

27. When $p = 4\pi\sqrt{M/k}$, the displacements of Example 6.33 become

$$\begin{aligned}
x(t) &= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1 - \cos\frac{4\pi}{p}(t - pn)\right]h(t - pn) \\
&= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1 - \cos\frac{4\pi t}{p}\right]h(t - np) \\
&= \frac{F}{k}\left(1 - \cos\frac{4\pi t}{p}\right)\sum_{n=0}^{\infty}(-1)^n h(t - np).
\end{aligned}$$

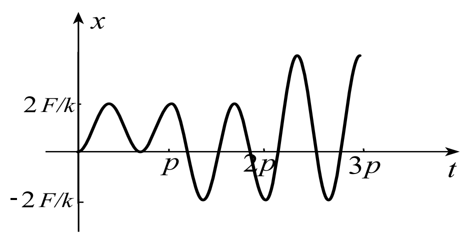
A graph of this funtion is shown below.



28. When $p = 3\pi\sqrt{M/k}$, the displacements of Example 6.33 become

$$\begin{aligned} x(t) &= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1-\cos\frac{3\pi}{p}(t-pn)\right]h(t-pn) \\ &= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1-(-1)^n\cos\frac{3\pi t}{p}\right]h(t-np) \\ &= \frac{F}{k}\sum_{n=0}^{\infty}\left[(-1)^n-\cos\frac{3\pi t}{p}\right)h(t-np). \end{aligned}$$

A graph of this funtion is shown below. There is resonance.



29. Since the force can be written algebraically as

$$20(t-1)[h(t-1)-h(t-2)]+20\,h(t-2) = 20(t-1)h(t-1)+(40-20t)h(t-2),$$

the initial-value problem for displacements of the mass is

$$\frac{d^2x}{dt^2}+16x = 20(t-1)h(t-1)+(40-20t)h(t-2), \qquad x(0)=0, \quad x'(0)=0.$$

When we take Laplace transforms,

$$s^2X+16X = 20e^{-s}\mathcal{L}\{t\}+e^{-2s}\mathcal{L}\{40-20(t+2)\} = \frac{20e^{-s}}{s^2}-\frac{20e^{-2s}}{s^2} = \frac{20(e^{-s}-e^{-2s})}{s^2}.$$

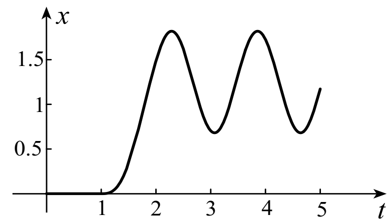
Solving for $X(s)$ gives

$$X(s) = \frac{20(e^{-s}-e^{-2s})}{s^2(s^2+16)} = 20(e^{-s}-e^{-2s})\left(\frac{1/16}{s^2}-\frac{1/16}{s^2+16}\right).$$

Inverse transforms now yield

$$\begin{aligned} x(t) &= \frac{5}{4}\left[(t-1)-\frac{1}{4}\sin 4(t-1)\right]h(t-1)-\frac{5}{4}\left[(t-2)-\frac{1}{4}\sin 4(t-2)\right]h(t-2) \\ &= \frac{5}{16}\left[4(t-1)-\sin 4(t-1)\right]h(t-1)-\frac{5}{16}\left[4(t-2)-\sin 4(t-2)\right]h(t-2)\text{ m.} \end{aligned}$$

A plot of this function is shown to the right.



EXERCISES 6.5

1. The initial-value problem governing the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = \sum_{n=1}^{\infty} 500\delta(t-60n) - \frac{5g}{10^6}, \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 = \sum_{n=1}^{\infty} 500e^{-60ns} - \frac{5G}{10^6},$$

from which

$$G(s) = \frac{5000}{s+5/10^6} + 500\sum_{n=1}^{\infty} \frac{e^{-60ns}}{s+5/10^6}.$$

Inverse transforms give

$$\begin{aligned} g(t) &= 5000e^{-5t/10^6} + 500\sum_{n=1}^{\infty} e^{-5(t-60n)/10^6} h(t-60n) \\ &= 500e^{-5t/10^6}\left[10 + \sum_{n=1}^{\infty} e^{3n/10^4} h(t-60n)\right]. \end{aligned}$$

2. The initial-value problem governing the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = \frac{1}{10} + \sum_{n=1}^{\infty} 500\delta(t-60n) - \frac{5g}{10^6}, \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 = \frac{10}{s} + \sum_{n=1}^{\infty} 500e^{-60ns} - \frac{5G}{10^6},$$

from which

$$G(s) = \frac{10/s + 5000}{s+5/10^6} + 500\sum_{n=1}^{\infty} \frac{e^{-60ns}}{s+5/10^6}.$$

Inverse transforms give

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left\{\frac{5000s+10}{s(s+5/10^6)}\right\} + 500\sum_{n=1}^{\infty} e^{-5(t-60n)/10^6} h(t-60n) \\ &= \mathcal{L}^{-1}\left\{\frac{10^7/5}{s} - \frac{399(10^4)}{s+5/10^6}\right\} + 500e^{-5t/10^6}\sum_{n=1}^{\infty} e^{3n/10^4} h(t-60n) \\ &= 200\,000 - 1\,995\,000e^{-5t/10^6} + 500e^{-5t/10^6}\sum_{n=1}^{\infty} e^{3n/10^4} h(t-60n). \end{aligned}$$

3. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 512} = \frac{1}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \frac{1}{32} \sin 16t \text{ m.}$$

4. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 80\frac{dx}{dt} + 512x = \delta(t), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 80[sX] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 80s + 512} = \frac{1}{2(s^2 + 40s + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2(s^2 + 40s + 256)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+20)^2 - 144}\right\} \\ &= \frac{e^{-20t}}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 144}\right\} = \frac{e^{-20t}}{2}\mathcal{L}^{-1}\left\{\frac{-1/24}{s+12} + \frac{1/24}{s-12}\right\} \\ &= \frac{e^{-20t}}{48}\left(-e^{-12t} + e^{12t}\right) = \frac{1}{48}\left(e^{-8t} - e^{-32t}\right) \text{ m.} \end{aligned}$$

5. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 512x = \delta(t), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 8[sX] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 8s + 512} = \frac{1}{2(s^2 + 4s + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2(s^2 + 4s + 256)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 + 252}\right\} \\ &= \frac{e^{-2t}}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 252}\right\} = \frac{e^{-2t}}{12\sqrt{7}} \sin 6\sqrt{7}t \text{ m.} \end{aligned}$$

6. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t - t_0), \qquad x(0) = x_0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X - x_0 s] + 512X = e^{-t_0 s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2x_0 s}{2s^2+512} + \frac{e^{-t_0 s}}{2s^2+512} = \frac{x_0 s}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0 s}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}\right\} = x_0\cos 16t + \frac{1}{32}\sin 16(t-t_0)\,h(t-t_0) \text{ m.}$$

7. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t-t_0), \qquad x(0)=0, \quad x'(0)=v_0.$$

When we take Laplace transforms,

$$2[s^2X - v_0] + 512X = e^{-t_0 s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2v_0}{2s^2+512} + \frac{e^{-t_0 s}}{2s^2+512} = \frac{v_0}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{v_0}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}\right\} = \frac{v_0}{16}\sin 16t + \frac{1}{32}\sin 16(t-t_0)\,h(t-t_0) \text{ m.}$$

8. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t-t_0), \qquad x(0)=x_0, \quad x'(0)=v_0.$$

When we take Laplace transforms,

$$2[s^2X - x_0 s - v_0] + 512X = e^{-t_0 s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2x_0 s + 2v_0}{2s^2+512} + \frac{e^{-t_0 s}}{2s^2+512} = \frac{x_0 s + v_0}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0 s + v_0}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}\right\} = x_0\cos 16t + \frac{v_0}{16}\sin 16t + \frac{1}{32}\sin 16(t-t_0)\,h(t-t_0) \text{ m.}$$

9. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \delta(t) + \delta(t-1), \qquad x(0)=0, \quad x'(0)=0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = 1 + e^{-s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1+e^{-s}}{s^2+100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1+e^{-s}}{s^2+100}\right\} = \frac{1}{10}\sin 10t + \frac{1}{10}\sin 10(t-1)\,h(t-1) \text{ m.}$$

10. When we take Laplace transforms,

$$Ms^2X + kX = 1 \qquad \Longrightarrow \qquad X(s) = \frac{1}{Ms^2 + k}.$$

The inverse transform is

$$x(t) = \frac{1}{M}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k/M}\right\} = \frac{1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}t.$$

This function satisfies the initial displacement condition $x(0) = 0$. The velocity of the mass is

$$v(t) = \frac{1}{\sqrt{kM}}\sqrt{\frac{k}{M}}\cos\sqrt{\frac{k}{M}}t = \frac{1}{M}\cos\sqrt{\frac{k}{M}}t.$$

The limit of this as $t \to 0$ is $1/M$. We have seen that an impulse force of size F causes a velocity change of F/M. To apply a unit impulse force at time $t = 0$ implies an instantaneous change of velocity $1/M$, and this is in conflict with the initial condition $x'(0) = 0$.

11. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \sum_{n=0}^{\infty}\delta(t-n), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = \sum_{n=0}^{\infty} e^{-ns}.$$

We solve this for the transform $X(s)$,

$$X(s) = \sum_{n=0}^{\infty}\frac{e^{-ns}}{s^2 + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^2+100}\right\} = \frac{1}{10}\sum_{n=0}^{\infty}\sin 10(t-n)\,h(t-n) \text{ m}.$$

12. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \sum_{n=0}^{\infty}\delta(t-n\pi/5), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = \sum_{n=0}^{\infty} e^{-n\pi s/5}.$$

We solve this for the transform $X(s)$,

$$X(s) = \sum_{n=0}^{\infty}\frac{e^{-n\pi s/5}}{s^2 + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{e^{-n\pi s/5}}{s^2+100}\right\} = \frac{1}{10}\sum_{n=0}^{\infty}\sin 10(t-n\pi/5)\,h(t-n\pi/5) \text{ m}.$$

To show that resonance occurs we rewrite the displacement in the form

$$x(t) = \frac{1}{10}\sum_{n=0}^{\infty} \sin(10t - 2n\pi)\, h(t - n\pi/5) = \frac{1}{10}\sin 10t \sum_{n=0}^{\infty} h(t - n\pi/5).$$

This shows that $x(t)$ becomes unbounded for large t.

13. The initial-value problem for displacement of the mass is

$$M\frac{d^2x}{dt^2} + kx = F\delta(t - t_0), \quad x(0) = x_0, \quad x'(0) = v_0.$$

When we apply Laplace transforms,

$$M[s^2X - x_0 s - v_0] + kX = Fe^{-t_0 s},$$

from which

$$X(s) = \frac{M(x_0 s + v_0)}{Ms^2 + k} + \frac{Fe^{-t_0 s}}{Ms^2 + k} = \frac{x_0 s + v_0}{s^2 + k/M} + \frac{F}{M}\left(\frac{e^{-t_0 s}}{s^2 + k/M}\right).$$

The inverse Laplace transform is

$$\begin{aligned} x(t) &= x_0 \cos\sqrt{\frac{k}{M}}t + v_0\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}t + \frac{F}{M}\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}(t - t_0)\, h(t - t_0) \\ &= x_0 \cos\sqrt{\frac{k}{M}}t + v_0\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}t + \frac{F}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(t - t_0)\, h(t - t_0). \end{aligned}$$

For the mass to come to an instantaneous stop at time $t = t_0$, we require

$$\begin{aligned} 0 = \lim_{t\to t_0^+} x'(t) &= \lim_{t\to t_0^+}\left[-x_0\sqrt{\frac{k}{M}}\sin\sqrt{\frac{k}{M}}t + v_0\cos\sqrt{\frac{k}{M}}t + \frac{F}{\sqrt{kM}}\sqrt{\frac{k}{M}}\cos\sqrt{\frac{k}{M}}(t - t_0)\, h(t - t_0)\right] \\ &= -x_0\sqrt{\frac{k}{M}}\sin\sqrt{\frac{k}{M}}t_0 + v_0\cos\sqrt{\frac{k}{M}}t_0 + \frac{F}{M}. \end{aligned}$$

We can solve this for F,

$$F = x_0\sqrt{kM}\sin\sqrt{\frac{k}{M}}t_0 - v_0 M\cos\sqrt{\frac{k}{M}}t_0.$$

14. (a) The initial-value problem for motion of the mass is

$$M\frac{d^2x}{dt^2} + kx = F\delta(t) - \mu Mg, \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms, we get

$$Ms^2X + kX = F - \frac{\mu Mg}{s},$$

from which

$$X(s) = \frac{F}{Ms^2 + k} - \frac{\mu Mg}{s(Ms^2 + k)} = \frac{F}{M}\left(\frac{1}{s^2 + k/M}\right) - \mu Mg\left(\frac{1/k}{s} - \frac{s/k}{s^2 + k/M}\right).$$

Inverse transforms now give

$$x(t) = \frac{F}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}t - \frac{\mu Mg}{k}\left(1 - \cos\sqrt{\frac{k}{M}}t\right).$$

Since the velocity of the mass is

$$x'(t) = \frac{F}{M}\cos\sqrt{\frac{k}{M}}t - \mu g\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}t,$$

the initial velocity of imparted to the mass is $\lim_{t\to 0^+} x'(t) = \dfrac{F}{M}$.

(b) The mass comes to a stop when its velocity is zero. This will be the smallest positive solution of the equation

$$0 = \frac{F}{M}\cos\sqrt{\frac{k}{M}}t - \mu g\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}t \quad\Longrightarrow\quad \tan\sqrt{\frac{k}{M}}t = \frac{F/M}{\mu g\sqrt{M/k}} = \frac{F\sqrt{k}}{\mu g M^{3/2}}.$$

Solutions of this equation are

$$t = \sqrt{\frac{M}{k}}\left[\text{Tan}^{-1}\left(\frac{F\sqrt{k}}{\mu g M^{3/2}}\right) + n\pi\right],$$

where n is an integer. The smallest positive solution is $t_0 = \sqrt{\dfrac{M}{k}}\text{Tan}^{-1}\left(\dfrac{F\sqrt{k}}{\mu g M^{3/2}}\right)$.

(c) The position of the mass at this time is

$$x(t_0) = \frac{F}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}t_0 - \frac{\mu M g}{k}\left(1 - \cos\sqrt{\frac{k}{M}}t_0\right).$$

Since $\tan\sqrt{\dfrac{k}{M}}t_0 = \dfrac{F\sqrt{k}}{\mu g M^{3/2}}$, it follows that

$$\sin\sqrt{\frac{k}{M}}t_0 = \frac{F\sqrt{k}}{\sqrt{F^2k + \mu^2 g^2 M^3}}, \qquad \cos\sqrt{\frac{k}{M}}t_0 = \frac{\mu g M^{3/2}}{\sqrt{F^2k + \mu^2 g^2 M^3}}.$$

With these,

$$\begin{aligned} x(t_0) &= \frac{F}{\sqrt{kM}}\frac{F\sqrt{k}}{\sqrt{F^2k + \mu^2 g^2 M^3}} - \frac{\mu M g}{k}\left(1 - \frac{\mu g M^{3/2}}{\sqrt{F^2k + \mu^2 g^2 M^3}}\right) \\ &= -\frac{\mu M g}{k} + \frac{1}{\sqrt{F^2k + \mu^2 g^2 M^3}}\left(\frac{F^2}{\sqrt{M}} + \frac{\mu^2 g^2 M^{5/2}}{k}\right) \\ &= -\frac{\mu M g}{k} + \frac{1}{k\sqrt{M}\sqrt{F^2k + \mu^2 g^2 M^3}}\left(F^2k + \mu^2 g^2 M^3\right) = -\frac{\mu M g}{k} + \frac{\sqrt{F^2k + \mu^2 g^2 M^3}}{k\sqrt{M}}. \end{aligned}$$

The spring force at this displacement is

$$x(t_0) = k\left(-\frac{\mu M g}{k} + \frac{\sqrt{F^2k + \mu^2 g^2 M^3}}{k\sqrt{M}}\right) = -\mu M g + \frac{\sqrt{F^2k + \mu^2 g^2 M^3}}{\sqrt{M}}.$$

Since the force of static friction is $\mu_s M g$, the mass moves to the left if

$$\begin{aligned} -\mu M g + \frac{\sqrt{F^2k + \mu^2 g^2 M^3}}{\sqrt{M}} &> \mu_s M g \\ F^2k + \mu^2 g^2 M^3 &> M^3 g^2(\mu_s + \mu)^2 \\ F^2k &> M^3 g^2(\mu_s^2 + 2\mu\mu_s) \\ F &> \sqrt{\frac{g^2 M^3 \mu_s(\mu_s + 2\mu)}{k}}. \end{aligned}$$

15. If we write $x(t)$ in the form

$$\begin{aligned} x(t) &= \frac{F}{ka}\left[\cos\sqrt{\frac{k}{M}}(t - t_0)\cos\sqrt{\frac{k}{M}}a + \sin\sqrt{\frac{k}{M}}(t - t_0)\sin\sqrt{\frac{k}{M}}a - \cos\sqrt{\frac{k}{M}}(t - t_0)\right] \\ &= \frac{F}{ka}\left[\left(\cos\sqrt{\frac{k}{M}}a - 1\right)\cos\sqrt{\frac{k}{M}}(t - t_0) + \sin\sqrt{\frac{k}{M}}a\sin\sqrt{\frac{k}{M}}(t - t_0)\right], \end{aligned}$$

the amplitude of the simple harmonic motion is

$$A = \frac{F}{ka}\sqrt{\left(\cos\sqrt{\frac{k}{M}}a - 1\right)^2 + \sin^2\sqrt{\frac{k}{M}}a} = \frac{F}{ka}\sqrt{2 - 2\cos\sqrt{\frac{k}{M}}a}$$
$$= \frac{\sqrt{2}F}{ka}\sqrt{1 - \left(1 - 2\sin^2\sqrt{\frac{k}{M}}\frac{a}{2}\right)} = \frac{2F}{ka}\left|\sin\sqrt{\frac{k}{M}}\frac{a}{2}\right|.$$

It is a question of whether $2\left|\sin\sqrt{\frac{k}{M}}\frac{a}{2}\right|$ can be less than one for some values of a and greater than one for other values of a. This is certainly possible. If a is such that $\sqrt{\frac{k}{M}}\frac{a}{2}$ is close to $\pi/2$, then $2\left|\sin\sqrt{\frac{k}{M}}\frac{a}{2}\right|$ will be greater than one, whereas if $\sqrt{\frac{k}{M}}\frac{a}{2}$ is close to zero, then $2\left|\sin\sqrt{\frac{k}{M}}\frac{a}{2}\right|$ will be less than one.

16. (a) With $v = 20$ km/hr or 50/9 m/s, the initial-value problem for displacement of the front end of the car is

$$200\frac{d^2y}{dt^2} + 1000y = f(50t/9), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(50t/9) = \begin{cases} \frac{3}{5}\left(\frac{50t}{9}\right)\left(1 - \frac{50t}{9}\right), & 0 < t < 9/50 \\ 0, & t > 9/50 \end{cases}$$
$$= \frac{10t}{27}(9 - 50t)\left[h(t) - h\left(t - \frac{9}{50}\right)\right].$$

When we take Laplace transforms,

$$200s^2Y + 1000Y = \frac{10}{27}\left[\left(\frac{9}{s^2} - \frac{100}{s^3}\right) - e^{-9s/50}\mathcal{L}\left\{\left(t + \frac{9}{50}\right)\left[9 - 50\left(t + \frac{9}{50}\right)\right]\right\}\right]$$
$$= \frac{10}{27}\left[\left(\frac{9}{s^2} - \frac{100}{s^3}\right) - e^{-9s/50}\mathcal{L}\left\{-9t - 50t^2\right\}\right]$$
$$= \frac{10}{27}\left[\left(\frac{9}{s^2} - \frac{100}{s^3}\right) + e^{-9s/50}\left(\frac{9}{s^2} + \frac{100}{s^3}\right)\right].$$

We now solve for $Y(s)$,

$$Y(s) = \frac{10}{27(200s^2 + 1000)}\left[\left(\frac{9}{s^2} - \frac{100}{s^3}\right) + e^{-9s/50}\left(\frac{9}{s^2} + \frac{100}{s^3}\right)\right]$$
$$= \frac{1}{540(s^2 + 5)}\left[\left(\frac{9s - 100}{s^3}\right) + e^{-9s/50}\left(\frac{9s + 100}{s^3}\right)\right].$$

Partial fractions give

$$Y(s) = \left[\frac{4}{s} + \frac{9}{5s^2} - \frac{20}{s^3} - \frac{20s + 9}{5(s^2 + 5)}\right] + e^{-9s/50}\left[-\frac{4}{s} + \frac{9}{5s^2} + \frac{20}{s^3} + \frac{20s - 9}{5(s^2 + 5)}\right].$$

Inverse transforms now yield

$$y(t) = 4 + \frac{9t}{5} - 10t^2 - 4\cos\sqrt{5}t - \frac{9\sqrt{5}}{25}\sin\sqrt{5}t + \left[-4 + \frac{9}{5}\left(t - \frac{9}{50}\right) + 10\left(t - \frac{9}{50}\right)^2\right.$$
$$\left. + 4\cos\sqrt{5}\left(t - \frac{9}{50}\right) - \frac{9\sqrt{5}}{25}\sin\sqrt{5}\left(t - \frac{9}{50}\right)\right]h\left(t - \frac{9}{50}\right).$$

(b) After the speed bump (when $t > 9/50$), the displacement of the car is

$$\begin{aligned}
y(t) &= 4 + \frac{9t}{5} - 10t^2 - 4\cos\sqrt{5}t - \frac{9\sqrt{5}}{25}\sin\sqrt{5}t + \left[-4 + \frac{9}{5}\left(t - \frac{9}{50}\right) + 10\left(t - \frac{9}{50}\right)^2\right.\\
&\quad \left. + 4\cos\sqrt{5}\left(t - \frac{9}{50}\right) - \frac{9\sqrt{5}}{25}\sin\sqrt{5}\left(t - \frac{9}{50}\right)\right]\\
&= -4\cos\sqrt{5}t - \frac{9\sqrt{5}}{25}\sin\sqrt{5}t + 4\cos\sqrt{5}\left(t - \frac{9}{50}\right) - \frac{9\sqrt{5}}{25}\sin\sqrt{5}\left(t - \frac{9}{50}\right)\\
&= -4\cos\sqrt{5}t - \frac{9\sqrt{5}}{25}\sin\sqrt{5}t + 4\left(\cos\sqrt{5}t\cos\frac{9\sqrt{5}}{50} + \sin\sqrt{5}t\sin\frac{9\sqrt{5}}{50}\right)\\
&\quad - \frac{9\sqrt{5}}{25}\left(\sin\sqrt{5}t\cos\frac{9\sqrt{5}}{50} - \cos\sqrt{5}t\sin\frac{9\sqrt{5}}{50}\right)\\
&= \left(-4 + 4\cos\frac{9\sqrt{5}}{50} + \frac{9\sqrt{5}}{25}\sin\frac{9\sqrt{5}}{50}\right)\cos\sqrt{5}t\\
&\quad + \left(-\frac{9\sqrt{5}}{25} + 4\sin\frac{9\sqrt{5}}{50} - \frac{9\sqrt{5}}{25}\cos\frac{9\sqrt{5}}{50}\right)\sin\sqrt{5}t.
\end{aligned}$$

This is simple harmonic motion with amplitude

$$\sqrt{\left(-4 + 4\cos\frac{9\sqrt{5}}{50} + \frac{9\sqrt{5}}{25}\sin\frac{9\sqrt{5}}{50}\right)^2 + \left(-\frac{9\sqrt{5}}{25} + 4\sin\frac{9\sqrt{5}}{50} - \frac{9\sqrt{5}}{25}\cos\frac{9\sqrt{5}}{50}\right)^2} = 0.022.$$

The amplitude is 2.2 centimetres.

17. When we take Laplace transforms of the differential equation, and use the initial conditions,

$$(s^2Y - s + 2) - 3(sY - 1) - 4Y = F(s) \qquad \Longrightarrow \qquad Y(s) = \frac{s-5}{s^2 - 3s - 4} + \frac{F(s)}{s^2 - 3s - 4}.$$

The transfer function is $H(s) = 1/(s^2 - 3s - 4)$, and the unit impulse response function is its inverse,

$$\hbar(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/5}{s-4} - \frac{1/5}{s+1}\right\} = \frac{1}{5}(e^{4t} - e^{-t}).$$

A general solution of the associated homogeneous problem is

$$y_h(t) = \mathcal{L}^{-1}\left\{\frac{s-5}{(s-4)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/5}{s-4} + \frac{6/5}{s+1}\right\} = \frac{1}{5}(6e^{-t} - e^{4t}).$$

Equation 6.38 gives the solution of the initial-value problem in the form

$$y(t) = \frac{1}{5}(6e^{-t} - e^{4t}) + \frac{1}{5}\int_0^t f(u)[6e^{-(t-u)} + e^{4(t-u)}]\,du.$$

18. When we take Laplace transforms of the differential equation, and use the initial conditions,

$$(s^2Y - As - B) + 2(sY - A) + 3Y = F(s) \qquad \Longrightarrow \qquad Y(s) = \frac{As + (B + 2A)}{s^2 + 2s + 3} + \frac{F(s)}{s^2 + 2s + 3}.$$

The transfer function is $H(s) = 1/(s^2 + 2s + 3)$, and the unit impulse response function is its inverse,

$$\hbar(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2}\right\} = \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t.$$

A general solution of the associated homogeneous problem is

$$y_h(t) = \mathcal{L}^{-1}\left\{\frac{As + (B + 2A)}{s^2 + 2s + 3}\right\} = \mathcal{L}^{-1}\left\{\frac{A(s + 1) + (A + B)}{(s + 1)^2 + 2}\right\}$$
$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{As + (A + B)}{s^2 + 2}\right\} = e^{-t}\left[A\cos\sqrt{2}t + \left(\frac{A + B}{\sqrt{2}}\right)\sin\sqrt{2}t\right].$$

Equation 6.38 gives the solution of the initial-value problem in the form

$$y(t) = e^{-t}\left[A\cos\sqrt{2}t + \left(\frac{A + B}{\sqrt{2}}\right)\sin\sqrt{2}t\right] + \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sin\sqrt{2}(t - u)\,du.$$

19. The unit impulse response function $\hbar(t)$ is the inverse transform of $1/P(s)$ which we would find by factoring $P(s)$. A real zero $s = a$ of $P(s)$ leads to a term e^{at} in $\hbar(t)$. It is bounded if, and only if, $a \le 0$. Complex conjugate zeros $a \pm bi$ lead to terms $e^{at}\cos bt$ and $e^{at}\sin at$. These are bounded if, and only if, $a \le 0$.

20. The unit impulse response function $\hbar(t)$ is the inverse transform of $1/P(s)$ which we would find by factoring $P(s)$. A real zero $s = a$ of $P(s)$ leads to a term e^{at} in $\hbar(t)$. It approaches zero if, and only if, $a < 0$. Complex conjugate zeros $a \pm bi$ lead to terms $e^{at}\cos bt$ and $e^{at}\sin at$. These approach zero if, and only if, $a < 0$.