The pattern emerging is that

$$d_n = 2^{n-1} - 2^{n-3} + 2^{n-4} - \cdots + (-1)^n.$$

If we multiply this by 2,

$$2d_n = 2^n - 2^{n-2} + 2^{n-3} - \cdots + 2(-1)^n,$$

and then add it to d_n,

$$3d_n = 2^n + 2^{n-1} - 2^{n-2} + (-1)^n.$$

Thus,

$$d_n = \frac{1}{3}\left[2^n + 2^{n-1} - 2^{n-2} + (-1)^n\right] = \frac{1}{3}\left[5 \cdot 2^{n-2} + (-1)^n\right].$$

Finally then, for $n \geq 2$,

$$c_n = \frac{d_n}{2^{n-2}} = \frac{1}{3 \cdot 2^{n-2}}\left[5 \cdot 2^{n-2} + (-1)^n\right] = \frac{5}{3} + \frac{(-1)^n}{3 \cdot 2^{n-2}}.$$

This formula also gives $c_1 = 1$.

EXERCISES 10.9

1. Since $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$, the series diverges by the n^{th} term test.

2. $\sum_{n=1}^{\infty} \frac{2^n}{5^{n+1}} = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$, a geometric series with sum $\frac{1}{5}\left(\frac{2/5}{1-2/5}\right) = \frac{2}{15}$.

3. Since $\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right) = 0 - 1 + 0 + 1 + 0 - 1 + 0 + 1 + \cdots$, terms do not approach zero, and the series diverges by the n^{th} term test.

4. Since $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$ (see expression 1.68), the series diverges by the n^{th} term test.

5. This is a geometric series with common ratio 49/9, and therefore the series diverges.

6. $\sum_{n=1}^{\infty} \frac{7^{n+3}}{3^{2n-2}} = \frac{7^3}{3^{-2}}\sum_{n=1}^{\infty}\left(\frac{7}{9}\right)^n$ is a geometric series with sum $7^3(3)^2\left(\frac{7/9}{1-7/9}\right) = \frac{21\,609}{2}$.

7. Since $\lim_{n\to\infty}\sqrt{\frac{n^2-1}{n^2+1}} = 1$, the series diverges by the n^{th} term test.

8. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$ is a geometric series with sum $\frac{-1/2}{1+1/2} = -\frac{1}{3}$.

9. Since terms of the series become arbitrarily large as n increases, the series diverges by the n^{th} term test.

10. Since $\lim_{n\to\infty} \mathrm{Tan}^{-1} n = \frac{\pi}{2}$, the series diverges by the n^{th} term test.

11. $0.666\,666\ldots = 0.6 + 0.06 + 0.006 + \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots = \frac{6/10}{1-1/10} = \frac{2}{3}$

12.
$$0.131\,313\,131\ldots = 0.13 + 0.001\,3 + 0.000\,013 + \cdots = \frac{13}{100} + \frac{13}{10\,000} + \frac{13}{1\,000\,000} + \cdots$$
$$= \frac{13/100}{1-1/100} = \frac{13}{99}$$

13. $1.347\,346\,346\,346\ldots = 1.347 + 0.000\,346 + 0.000\,000\,346 + \cdots = \dfrac{1347}{1000} + \dfrac{346}{10^6} + \dfrac{346}{10^9} + \cdots$

$$= \frac{1347}{1000} + \frac{346/10^6}{1-1/10^3} = \frac{1\,345\,999}{999\,000}$$

14. $43.020\,502\,050\,205\ldots = 43 + 0.020\,5 + 0.000\,002\,05 + \cdots = 43 + \dfrac{205}{10^4} + \dfrac{205}{10^8} + \cdots$

$$= 43 + \frac{205/10^4}{1-1/10^4} = \frac{430\,162}{9999}$$

15. If $\sum c_n$ and $\sum d_n$ converge, then $\sum(c_n + d_n)$ converges.
Proof: Let $\{C_n\}$ and $\{D_n\}$ be the sequences of partial sums for $\sum c_n$ and $\sum d_n$ with limits C and D. The sequence of partial sums for $\sum(c_n + d_n)$, is $\{C_n + D_n\}$. According to part (ii) of Theorem 10.10, this sequence has limit $C + D$. Consequently, $\sum(c_n + d_n)$ converges to $C + D$.

16. If $\sum c_n$ converges and $\sum d_n$ diverges, then $\sum(c_n + d_n)$ diverges.
Proof: Assume to the contrary that $\sum(c_n + d_n)$ converges. Let $\{C_n\}$ and $\{D_n\}$ be the sequences of partial sums for $\sum c_n$ and $\sum d_n$. It follows that $\lim_{n\to\infty} C_n$ exists, call it C, but $\lim_{n\to\infty} D_n$ does not exist. $\{C_n + D_n\}$ is the sequence of partial sums for $\sum(c_n + d_n)$, and by assumption, it has a limit, call it E. But then according to part (ii) of Theorem 10.10, the sequence $\{(C_n + D_n) - C_n\} = \{D_n\}$ must have limit $E - C$, a contradiction. Consequently, our assumption that $\sum(c_n + d_n)$ converges must be incorrect.

17. If $\sum c_n$ and $\sum d_n$ diverge, then $\sum(c_n + d_n)$ may converge or diverge.
Proof: We give an example of each situation. The series $\sum n$ and $\sum(-n)$ both diverge, but their sum $\sum(n-n) = \sum 0$ has sum 0. On the other hand, the sum of $\sum n$ and $\sum n$ is $\sum 2n$ which diverges.

18. Since $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{4^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{4^n}$ are both geometric series with sums

$$\sum_{n=1}^{\infty}\frac{2^n}{4^n} = \frac{1/2}{1-1/2} = 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{3^n}{4^n} = \frac{3/4}{1-3/4} = 3,$$

then, by Exercise 15, $\displaystyle\sum_{n=1}^{\infty}\frac{2^n + 3^n}{4^n} = 1 + 3 = 4.$

19. Since $\displaystyle\sum_{n=1}^{\infty}(3/2)^n$ is a divergent geometric series, and $\displaystyle\sum_{n=1}^{\infty}(1/2)^n$ is a convergent geometric series, it follows from Exercise 16, that the given series diverges. (It also diverges by the n^{th} term test.)

20. Since $\displaystyle\lim_{n\to\infty}\frac{n^2 + 2^{2n}}{4^n} = \lim_{n\to\infty}\left(\frac{n^2}{4^n} + 1\right) = 1$, the series diverges by the n^{th} term test.

21. Since $\displaystyle\lim_{n\to\infty}\frac{2^n + 4^n - 8^n}{2^{3n}} = \lim_{n\to\infty}\left(\frac{1}{2^{2n}} + \frac{1}{2^n} - 1\right) = -1$, the series diverges by the n^{th} term test.

22. Since $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, the n^{th} partial sum of the series is

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = 1$, it follows that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1.$

23. The total distance travelled is $\displaystyle 20 + \sum_{n=1}^{\infty} 40(0.99)^n$. The series is geometric with sum $20 + \dfrac{40(0.99)}{1-0.99} = 3980$ m.

24. The total time taken to come to rest is

$$\sqrt{\frac{40}{9.81}} + t_1 + t_2 + t_3 + \cdots = \sqrt{\frac{40}{9.81}} + \sum_{n=1}^{\infty} t_n = \sqrt{\frac{40}{9.81}} + \sum_{n=1}^{\infty} \frac{4}{\sqrt{0.981}}(0.99)^{n/2}$$
$$= \sqrt{\frac{40}{9.81}} + \frac{4\sqrt{0.99}/\sqrt{0.981}}{1-\sqrt{0.99}} = 804 \text{ s.}$$

25. The total distance run by the dog is $\displaystyle \frac{2}{3} + \sum_{n=1}^{\infty} \frac{8}{3^{n+1}} = \frac{2}{3} + \frac{8/9}{1-1/3} = 2$ km.

We could also have reasoned this without series. Since the dog runs twice as fast as the farmer, and the farmer walks 1 km, the dog must run 2 km.

26. According to Exercise 10.1–61,

$$A_n = \frac{\sqrt{3}P^2}{36}\left(1 + \frac{1}{3} + \frac{4}{3^3} + \frac{4^2}{3^5} + \cdots + \frac{4^{n-1}}{3^{2n-1}}\right) \quad \text{(a finite geometric series after first term)}$$
$$= \frac{\sqrt{3}P^2}{36}\left\{1 + \frac{(1/3)[1-(4/9)^n]}{1-4/9}\right\} \quad \text{(using 10.39a)}$$
$$= \frac{\sqrt{3}P^2}{180}\left[8 - 3\left(\frac{4}{9}\right)^n\right].$$

Thus, $\displaystyle \lim_{n\to\infty} A_n = \frac{\sqrt{3}P^2}{180}(8) = \frac{2\sqrt{3}P^2}{45}$.

27. The inequality is certainly true for $x \ge 0$ and any n. To discuss the case when $x < 0$, we sum the geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

When $x < 0$ and n is even, then $1 - x^{n+1} > 0$ and $1 - x > 0$. Hence, $(1-x^{n+1})/(1-x) > 0$. When n is odd, and $-1 \le x < 0$, then $1 - x^{n+1} \ge 0$ and $1 - x > 0$. Hence, $(1-x^{n+1})/(1-x) > 0$. Finally, when n is odd, and $x < -1$, then $1 - x^{n+1} < 0$ and $1 - x > 0$. Hence, $(1-x^{n+1})/(1-x) < 0$. Consequently, the inequality is valid for all x when n is even, and for $x \ge -1$ when n is odd.

28. (a) If we subtract $S_n = 1 + r + r^2 + \cdots + r^{n-1}$ from $T_n = 1 + 2r + 3r^2 + \cdots + nr^{n-1}$, we obtain

$$T_n - S_n = r + 2r^2 + 3r^3 + \cdots + (n-1)r^{n-1} = r[1 + 2r + 3r^2 + \cdots + (n-1)r^{n-2}] = r(T_n - nr^{n-1}).$$

When we solve this for T_n and substitute for S_n,

$$T_n = \frac{S_n - nr^n}{1-r} = \frac{\dfrac{1-r^n}{1-r} - nr^n}{1-r} = \frac{1 - r^n - nr^n + nr^{n+1}}{(1-r)^2} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}.$$

If we now take limits as $n \to \infty$, we obtain

$$\sum_{n=1}^{\infty} nr^{n-1} = \lim_{n\to\infty} \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2} = \frac{1}{(1-r)^2}, \quad \text{provided } |r| < 1.$$

(b) If we set $\displaystyle S(r) = \sum_{n=1}^{\infty} nr^{n-1}$, and integrate with respect to r,

$$\int S(r)\,dr + C = \sum_{n=1}^{\infty} r^n = \frac{r}{1-r}.$$

Differentiation now gives $\displaystyle S(r) = \frac{(1-r)(1) - r(-1)}{(1-r)^2} = \frac{1}{(1-r)^2}$.

29. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots=\frac{1}{2}\left(1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\cdots\right)=\frac{1}{2}\left[\frac{1}{(1-1/2)^2}\right]=2$

30. $\frac{2}{5}+\frac{4}{25}+\frac{6}{125}+\frac{8}{625}+\cdots=\frac{2}{5}\left(1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\cdots\right)=\frac{2/5}{(1-1/5)^2}=\frac{5}{8}$

31. $\frac{2}{3}+\frac{3}{27}+\frac{4}{243}+\frac{5}{2187}+\cdots=3\left(1+\frac{2}{9}+\frac{3}{81}+\frac{4}{729}+\cdots\right)-3=3\left[\frac{1}{(1-1/9)^2}\right]-3=\frac{51}{64}$

32. $\frac{12}{5}+\frac{48}{25}+\frac{192}{125}+\frac{768}{625}+\cdots=\frac{12}{5}\left(1+\frac{4}{5}+\frac{16}{25}+\frac{64}{125}+\cdots\right)=\frac{12/5}{1-4/5}=12$

33. The probability that the first person wins on the first toss is $1/2$. The probability that the first person wins on the second toss is the product of the following three probabilities:

probability that first person throws a tail on the first toss $= 1/2$;
probability that second person throws a tail on first toss $= 1/2$;
probability that first person throws a head on second toss $= 1/2$.

The resultant probabilty is $(1/2)(1/2)(1/2) = 1/2^3$. The probability that the first person wins on the third toss is the product of the following five probabilities:

probability that first person throws a tail on the first toss $= 1/2$;
probability that second person throws a tail on first toss $= 1/2$;
probability that first person throws a tail on second toss $= 1/2$.
probability that second person throws a tail on the second toss $= 1/2$;
probability that first person throws a head on third toss $= 1/2$;

The resultant probabilty is $1/2^5$.
Continuation of this process leads to the following probability that the first person to toss wins

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots=\frac{1/2}{1-1/4}=\frac{2}{3}.$$

34. The probability that the first person wins on the first toss is $1/6$. The probability that the first person wins on the second toss is the product of the following three probabilities:

probability that first person does not throw a six on the first toss $= 5/6$;
probability that second person does not throw a six on first toss $= 5/6$;
probability that first person throws a six on second toss $= 1/6$.

The resultant probabilty is $(5/6)(5/6)(1/6) = 5^2/6^3$. The probability that the first person wins on the third toss is the product of the following five probabilities:

probability that first person does not throw a six on the first toss $= 5/6$;
probability that second person does not throw a six on first toss $= 5/6$;
probability that first person does not throw a six on second toss $= 5/6$.
probability that second person does not throw a six on the second toss $= 5/6$;
probability that first person throws a six on third toss $= 1/6$;

The resultant probabilty is $5^4/6^5$.
Continuation of this process leads to the following probability that the first person to toss wins

$$\frac{1}{6}+\frac{5^2}{6^3}+\frac{5^4}{6^5}+\frac{5^6}{6^7}+\cdots=\frac{1/6}{1-25/36}=\frac{6}{11}.$$

35. Since the radius of convergence of the series is $R=\lim_{n\to\infty}\left|\frac{1/2^n}{1/2^{n+1}}\right|=2$, the open interval of convergence is $-2<x<2$. At $x=2$, the power series reduces to $\sum_{n=0}^{\infty}1$ which diverges by the n^{th} term test. At $x=-2$, it reduces to $\sum_{n=0}^{\infty}(-1)^n$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-2<x<2$.

36. Since the radius of convergence of the series is $R=\lim_{n\to\infty}\left|\frac{n^2 3^n}{(n+1)^2 3^{n+1}}\right|=1/3$, the open interval of convergence is $-1/3<x<1/3$. At $x=1/3$, the power series reduces to $\sum_{n=1}^{\infty}n^2$ which diverges by the n^{th} term test. At $x=-1/3$, it reduces to $\sum_{n=1}^{\infty}(-1)^n n^2$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-1/3<x<1/3$.

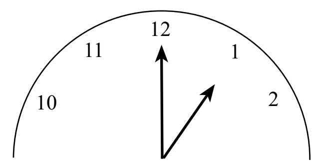
37. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{2^n \left(\dfrac{n-1}{n+1}\right)^2}{2^{n+1}\left(\dfrac{n}{n+2}\right)^2} \right| = 1/2$, the open interval of convergence is $7/2 < x < 9/2$. At $x = 9/2$, the power series reduces to $\sum_{n=2}^{\infty} (n-1)^2/(n+1)^2$ which diverges by the n^{th} term test. At $x = 7/2$, it reduces to $\sum_{n=2}^{\infty} (-1)^n (n-1)^2/(n+1)^2$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $7/2 < x < 9/2$.

38. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty} \left| \dfrac{(-1)^n}{(-1)^{n+1}} \right| = 1$, the radius of convergence of the given series is $R_x = 1$. The open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=0}^{\infty} (-1)^n$ which diverges by the n^{th} term test. At $x = -1$, it reduces to $\sum_{n=0}^{\infty} 1$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-1 < x < 1$.

39. While Achilles makes up the head start L, the tortoise moves a further distance L/c. While Achilles makes up this distance, the tortoise moves a further distance $(L/c)/c = L/c^2$. Continuation of this process gives the following distance traveled by Achilles in catching the tortoise

$$L + \frac{L}{c} + \frac{L}{c^2} + \frac{L}{c^3} + \cdots = \frac{L}{1 - 1/c} = \frac{cL}{c-1}.$$

40. (a) The minute hand moves 12 times as fast as the hour hand. While the minute hand moves through the angle $\pi/6$ radians from 12 at 1:00 to 1 at 1:05, the hour hand moves a further $(\pi/6)/12$ radians. While the minute hand moves through this angle, the hour hand moves through a further angle $[(\pi/6)/12]/12 = (\pi/6)/12^2$. Continuation of this process leads to the following angle traveled by the minute hand in catching the hour hand

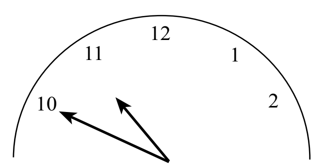


$$\frac{\pi}{6} + \frac{\pi/6}{12} + \frac{\pi/6}{12^2} + \frac{\pi/6}{12^3} + \cdots = \frac{\pi/6}{1 - 1/12} = \frac{2\pi}{11}.$$

This angle represents $\dfrac{2\pi}{11}\left(\dfrac{60}{2\pi}\right) = \dfrac{60}{11}$ minutes after 1:00.

(b) If we take time $t = 0$ at 1:00, the angle θ through which the minute hand moves in time t (in minutes) is $\theta = 2\pi t/60$. The angle ϕ that the hour hand makes with the vertical is $\phi = 2\pi t/720 + \pi/6$. These angles will be the same when $\dfrac{2\pi t}{60} = \dfrac{2\pi t}{720} + \dfrac{\pi}{6}$, the solution of which is $60/11$ minutes.

41. (a) The minute hand moves 12 times as fast as the hour hand. While the minute hand moves through the angle $5\pi/3$ radians from 12 at 10:00 to 10 at 10:50, the hour hand moves a further $(5\pi/3)/12$ radians. While the minute hand moves through this angle, the hour hand moves through a further angle $[(5\pi/3)/12]/12 = (5\pi/3)/12^2$. Continuation of this process leads to the following angle traveled by the minute hand in catching the hour hand



$$\frac{5\pi}{3} + \frac{5\pi/3}{12} + \frac{5\pi/3}{12^2} + \frac{5\pi/3}{12^3} + \cdots = \frac{5\pi/3}{1 - 1/12} = \frac{20\pi}{11}.$$

This angle represents $\dfrac{20\pi}{11}\left(\dfrac{60}{2\pi}\right) = \dfrac{600}{11}$ minutes after 10:00.

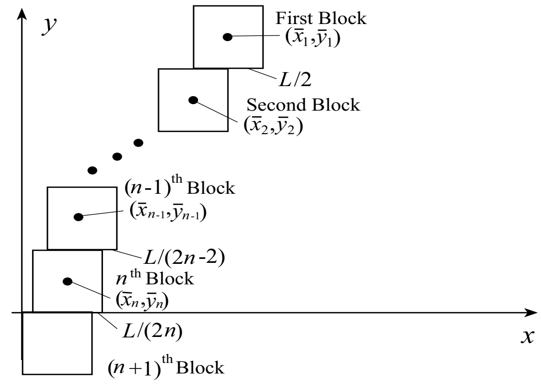
(b) If we take time $t = 0$ at 10:00, the angle θ through which the minute hand moves in time t (in minutes) is $\theta = 2\pi t/60$. The angle ϕ that the hour hand makes with the vertical is $\phi = 2\pi t/720 + 5\pi/3$. These angles will be the same when $\dfrac{2\pi t}{60} = \dfrac{2\pi t}{720} + \dfrac{5\pi}{3}$, the solution of which is $600/11$ minutes.

42. Suppose the length of each block is L. Taking the density of the blocks as unity, the mass of the top n blocks is nL^3.
The first moment of the n^{th} block about the y-axis is

$$L^3\overline{x}_n = L^3\left(\frac{L}{2}+\frac{L}{2n}\right) = \frac{L^4}{2}\left(1+\frac{1}{n}\right).$$

The first moment of the $(n-1)^{\text{th}}$ block about the y-axis is

$$L^3\overline{x}_{n-1} = L^3\left[\frac{L}{2}+\frac{L}{2n}+\frac{L}{2(n-1)}\right] = \frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}\right).$$



Continuing in this way, the moment of the first block about the y-axis is

$$L^3\overline{x}_1 = L^3\left[\frac{L}{2}+\frac{L}{2n}+\frac{L}{2(n-1)}+\cdots+\frac{L}{2}\right] = \frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right).$$

The x-coordinate of the centre of mass of the top n blocks is therefore

$$\begin{aligned}\overline{x} &= \frac{1}{nL^3}\left[\frac{L^4}{2}\left(1+\frac{1}{n}\right)+\frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}\right)+\cdots+\frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right)\right]\\ &= \frac{L}{2n}\left[n(1)+n\left(\frac{1}{n}\right)+(n-1)\left(\frac{1}{n-1}\right)+\cdots+2\left(\frac{1}{2}\right)+1(1)\right] = \frac{L}{2n}(2n) = L.\end{aligned}$$

Thus, the centre of mass of the top n blocks is over the edge of the $(n+1)^{\text{th}}$ block. They will not tip, but they are in a state of precarious equilibrium.

The right edge of the top block sticks out the following distance over the right edge of the $(n+1)^{\text{th}}$ block

$$\frac{L}{2}+\frac{L}{4}+\frac{L}{6}+\cdots+\frac{L}{2n} = \frac{L}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

This is $L/2$ times the n^{th} partial sum of the harmonic series which we know becomes arbitrarily large as n increases. Hence, the top n blocks can be made to protrude arbitrarily far over the $(n+1)^{\text{th}}$ block.

43. Let $\{S_n\}$ be the sequence of partial sums of the given series. It converges to the sum of the series, call it S. If terms of the series are grouped together, then the sequence of partial sums of the new series, call it $\{T_n\}$, is a subsequence of $\{S_n\}$. But every subseqeuence of a convergent series must converge to the same limit as the sequence. Thus, $\{T_n\}$ converges to S also, and the grouped series has sum S.

44. To verify this, we first write the Laplace transform as an infinite series of integrals

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \sum_{n=0}^\infty \int_{np}^{(n+1)p} e^{-st}f(t)\,dt.$$

If we change variables of integration in the n^{th} term with $u = t - np$, then

$$F(s) = \sum_{n=0}^\infty \int_0^p e^{-s(u+np)}f(u+np)\,du = \sum_{n=0}^\infty e^{-nps}\int_0^p e^{-su}f(u)\,du = \left(\int_0^p e^{-su}f(u)\,du\right)\left(\sum_{n=0}^\infty e^{-nps}\right).$$

Since the series is geometric with common ratio e^{-ps},

$$F(s) = \int_0^p e^{-su}f(u)\,du\left[\frac{1}{1-e^{-ps}}\right] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt.$$

45. (a) When V is the voltage across the capacitor, and resistor R_2 (they are in parallel), the currents through these devices are $i_C = CdV/dt$ and $i_{R_2} = V/R_2$. The current through R_1 must be the sum of these, $i_{R_1} = V/R_2 + CdV/dt$. The voltage across R_1 is therefore $R_1(V/R_2 + CdV/dt)$, and it follows that for $V_{\text{in}} = \overline{V}$,

$$\overline{V} = V + R_1\left(\frac{V}{R_2} + C\frac{dV}{dt}\right) \quad\Longrightarrow\quad \frac{dV}{dt} + \tau V = \alpha\overline{V},$$

where $\tau = (R_1 + R_2)/(R_1R_2C)$ and $\alpha = 1/(R_1C)$.

(b) If we multiply the differential equation by $e^{\tau t}$, the left side becomes the derivative of a product,

$$e^{\tau t}\frac{dV}{dt} + \tau e^{\tau t}V = \alpha\overline{V}e^{\tau t} \Longrightarrow \frac{d}{dt}(Ve^{\tau t}) = \alpha\overline{V}e^{\tau t} \Longrightarrow Ve^{\tau t} = \frac{\alpha\overline{V}}{\tau}e^{\tau t} + D \Longrightarrow V = \frac{\alpha\overline{V}}{\tau} + De^{-\tau t}.$$

Using the condition that $\lim_{t\to 2(n-1)T^+} V(t) = V_{n-1}$, we obtain

$$V_{n-1} = \frac{\alpha\overline{V}}{\tau} + De^{-2\tau(n-1)T} \quad\Longrightarrow\quad D = \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{2\tau(n-1)T}.$$

Hence, for $2(n-1)T < t < (2n-1)T$,

$$V(t) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{2\tau(n-1)T}e^{-\tau t} = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau[t-2(n-1)T]}.$$

At $t = (2n-1)T$,

$$V\big((2n-1)T\big) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau[(2n-1)T-2(n-1)T]} = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}.$$

(c) When $V_{\text{in}} = 0$, the rectifier prevents the charge that has been stored in the capacitor from flowing back through R_1; it simply discharges itself through R_2. Consequently, $dV/dt + \sigma V = 0$ where $\sigma = 1/(R_2C)$.

(d) We separate the differential equation:

$$\frac{dV}{V} = -\sigma\,dt \qquad\Longrightarrow\qquad \ln|V| = -\sigma t + D \qquad\Longrightarrow\qquad V(t) = Ee^{-\sigma t}.$$

If we now use the fact that $\lim_{t\to(2n-1)T^+} V(t) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}$, we obtain

$$\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T} = Ee^{-\sigma(2n-1)T} \quad\Longrightarrow\quad E = \frac{\alpha\overline{V}}{\tau}e^{\sigma(2n-1)T} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-[\tau-\sigma(2n-1)]T}.$$

Hence, for $(2n-1)T < t < 2nT$, we have $\quad V(t) = \left[\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}\right]e^{-\sigma[t-(2n-1)T]}.$

(e) When the function in (d) is evaluated at $t = 2nT$, its value is V_n; that is,

$$V_n = \left[\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}\right]e^{-\sigma T} = pV_{n-1} + q,$$

where $p = e^{-T(\tau+\sigma)}$ and $q = (\alpha\overline{V}/\tau)(1 - e^{-\tau T})e^{-\sigma T}$. If we iterate this recursive definition,

$$V_1 = pV_0 + q, \quad V_2 = pV_1 + q = p^2V_0 + q(p+1), \quad V_3 = pV_2 + q = p^3V_0 + q(p^2 + p + 1).$$

The pattern emerging is $V_n = p^nV_0 + q(1 + p + p^2 + \cdots + p^{n-1}) = p^nV_0 + \dfrac{q(1-p^n)}{1-p}$.

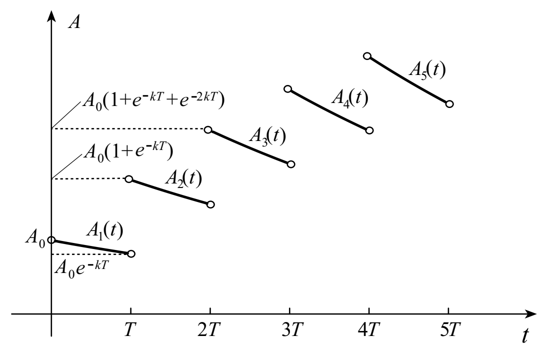
Since $V_0 = 0$, if the voltage across the capacitor is zero at time $t = 0$, we have

$$V_n = \frac{q(1-p^n)}{1-p} = \frac{\alpha\overline{V}}{\tau}(1 - e^{-\tau T})e^{-\sigma T}\left[\frac{1 - e^{-nT(\tau+\sigma)}}{1 - e^{-T(\tau+\sigma)}}\right].$$

46. (a) After time t, the amount of the first injection remaining is A_0e^{-kt}; the amount of the second injection remaining is $A_0e^{-k(t-T)}$; the amount of the third injection remaining is $A_0e^{-k(t-2T)}$; etc. At time t between the n^{th} and $(n+1)^{\text{th}}$ injection, the total amount remaining is

$$\begin{aligned} A_n(t) &= A_0e^{-kt} + A_0e^{-k(t-T)} + \cdots + A_0e^{-k[t-(n-1)T]} \\ &= A_0e^{-kt}\left[1 + e^{kT} + e^{2kT} + \cdots + e^{(n-1)kT}\right] \\ &= A_0e^{-kt}\left[\frac{1-(e^{kT})^n}{1-e^{kT}}\right] \qquad \text{(using 10.39a)} \\ &= A_0e^{-kt}\left[\frac{1-e^{knT}}{1-e^{kT}}\right] \qquad (n-1)T < t < nT. \end{aligned}$$

(b)



(c) $$\lim_{n\to\infty} A_n[(n-1)T] = \lim_{n\to\infty} A_0e^{-k(n-1)T}\left[\frac{1-e^{knT}}{1-e^{kT}}\right]$$
$$= \frac{A_0e^{kT}}{1-e^{kT}}\lim_{n\to\infty}(e^{-knT}-1) = \frac{-A_0e^{kT}}{1-e^{kT}} = \frac{A_0}{1-e^{-kT}}$$

EXERCISES 10.10

1. Since $l = \lim_{n\to\infty}\dfrac{\frac{1}{2n+1}}{\frac{1}{2n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

2. Since $l = \lim_{n\to\infty}\dfrac{\frac{1}{4n-3}}{\frac{1}{4n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

3. Since $l = \lim_{n\to\infty}\dfrac{\frac{1}{2n^2+4}}{\frac{1}{2n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

4. Since $l = \lim_{n\to\infty}\dfrac{\frac{1}{5n^2-3n-1}}{\frac{1}{5n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5n^2} = \frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).