# CHAPTER 10

## EXERCISES 10.1

**1.** This sequence has limit 0.

**2.** This sequence diverges.

**3.** This sequence has limit 3.

**4.** This sequence has limit 0.

**5.** This sequence diverges.

**6.** This sequence has limit 0.

**7.** This sequence diverges.

**8.** This sequence has limit $\lim_{n\to\infty} \frac{n}{n^2+n+2} = \lim_{n\to\infty} \frac{1}{n+1+2/n} = 0.$

**9.** This sequence has limit 0.

**10.** This sequence has limit $\pi/2$.

**11.** This sequence has limit 0.

**12.** This sequence diverges.

**13.** This sequence has limit 2 (since all terms are equal to 2).

**14.** This sequence has limit 0.

**15.** This sequence has limit $\lim_{n\to\infty} \frac{n+1}{2n+3} = \lim_{n\to\infty} \frac{1+1/n}{2+3/n} = \frac{1}{2}.$

**16.** This sequence has limit $\lim_{n\to\infty} \frac{2n+3}{n^2-5} = \lim_{n\to\infty} \frac{2+3/n}{n-5/n} = 0.$

**17.** This sequence has limit $\lim_{n\to\infty} \frac{n^2+5n-4}{n^2+2n-2} = \lim_{n\to\infty} \frac{1+5/n-4/n^2}{1+2/n-2/n^2} = 1.$

**18.** This sequence has limit 0.

**19.** This sequence has limit 0.

**20.** This sequence has limit $\lim_{n\to\infty} \frac{1}{1+1/n} \mathrm{Tan}^{-1} n = \frac{\pi}{2}.$

**21.** The general term is $\frac{2^n-1}{2^n}$.

**22.** The general term is $\frac{3n+1}{n^2}$.

**23.** The general term is $(-1)^{n+1}\frac{\ln(n+1)}{\sqrt{n+1}}$.

**24.** The general term is $\frac{1+(-1)^{n+1}}{2}$.

**25.** The general term is $\sqrt{2}\sin\frac{(2n-1)\pi}{4}$.

**26.** The limit of the sequence $\{\ln n/\sqrt{n}\}$ as $n\to\infty$ is equal to the limit of the function $\ln x/\sqrt{x}$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule on the limit of the function,

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

**27.** The limit of the sequence $\{(n^3+1)/e^n\}$ as $n\to\infty$ is equal to the limit of the function $(x^3+1)/e^x$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule on the limit of the function,

$$\lim_{n\to\infty} \frac{n^3+1}{e^n} = \lim_{x\to\infty} \frac{x^3+1}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

**28.** The limit of the sequence $\{n\sin(4/n)\}$ as $n\to\infty$ is equal to the limit of the function $x\sin(4/x)$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule,

$$\lim_{n\to\infty} n\sin\left(\frac{4}{n}\right) = \lim_{x\to\infty} x\sin\left(\frac{4}{x}\right) = \lim_{x\to\infty} \frac{\sin(4/x)}{1/x} = \lim_{x\to\infty} \frac{-(4/x^2)\cos(4/x)}{-1/x^2} = 4.$$

**29.** The limit of the sequence $\{[(n+5)/(n+3)]^n\}$ as $n \to \infty$ is equal to the limit of the function $[(x+5)/(x+3)]^x$ as $x \to \infty$, provided the limit of the function exists. We set $L$ equal to the limit of the function, take logarithms, and use L'Hôpital's rule,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty} x\ln\left(\frac{x+5}{x+3}\right) = \lim_{x\to\infty}\frac{\ln\left(\dfrac{x+5}{x+3}\right)}{1/x}$$

$$= \lim_{x\to\infty}\frac{\dfrac{x+3}{x+5}\left[\dfrac{(x+3)-(x+5)}{(x+3)^2}\right]}{-1/x^2} = \lim_{x\to\infty}\frac{2x^2}{(x+3)(x+5)} = 2.$$

Thus, $L = e^2$, and this is also the limit of the sequence.

**30.** Certainly the sequence diverges; terms get arbitrarily large for large $n$. On the other hand, as $n$ increases, the difference between terms approaches $\lim_{n\to\infty}[\ln n - \ln(n+1)] = \lim_{n\to\infty}\ln\left(\dfrac{n}{n+1}\right) = 0$.

**31.** (a) The first ten terms are 2,3,5,7,11,13,17,19,23,29. (b) No one has developed a formula for all primes.

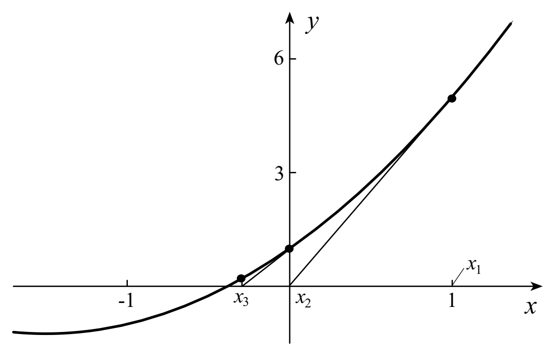
**32.** The figure indicates that with initial approximation $x_1 = 1$, the sequence defined by Newton's iterative procedure has a limit near $-1/2$. Iteration of

$$x_1 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$$x_2 = 0, \qquad x_3 = -1/3,$$
$$x_4 = -0.381, \qquad x_5 = -0.381\,966,$$
$$x_6 = -0.381\,966\,01, \qquad x_7 = -0.381\,966\,01.$$

Since $f(-0.381\,965\,95) = 1.4\times 10^{-7}$ and $f(-0.381\,966\,05) = -8.7\times 10^{-8}$, we can say that to seven decimals $x = -0.381\,966\,0$.




**33.** The figure indicates that with initial approximation $x_1 = -1$, the sequence defined by Newton's iterative procedure has a limit near $-1/2$. Iteration of

$$x_1 = -1, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$$x_2 = 0, \qquad x_3 = -1/3,$$
$$x_4 = -0.381, \qquad x_5 = -0.381\,966,$$
$$x_6 = -0.381\,966\,01, \qquad x_7 = -0.381\,966\,01.$$

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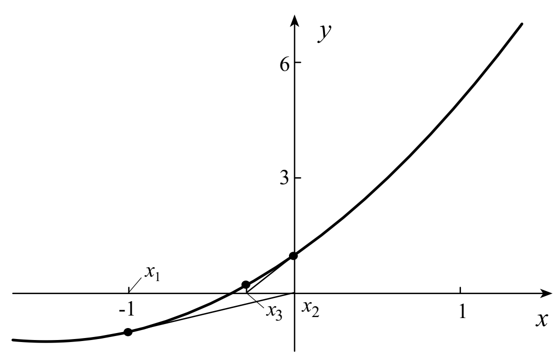
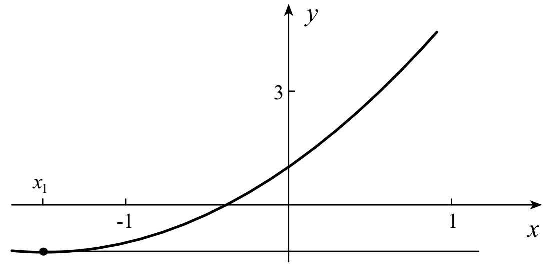



**34.** The figure indicates that with initial approximation $x_1 = -1.5$, the sequence defined by Newton's iterative procedure does not have a limit. This is because $x_1 = -1.5$ is a critical point of the function.



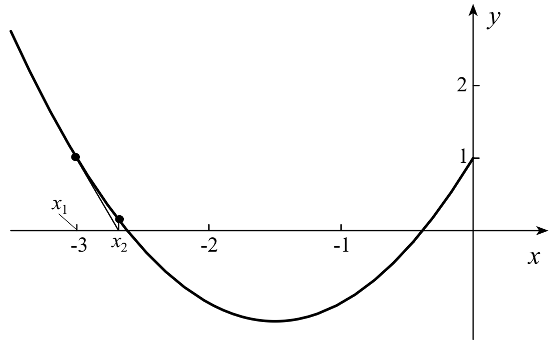
**35.** The figure indicates that with initial approximation $x_1 = -3$, the sequence defined by Newton's iterative procedure has a limit near $-3$. Iteration of

$$x_1 = -3, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$$x_2 = -2.667, \qquad x_3 = -2.6191,$$
$$x_4 = -2.618\,034\,5, \qquad x_5 = -2.618\,033\,99,$$
$$x_6 = -2.618\,033\,99.$$

Since $f(-2.618\,034\,05) = 1.4 \times 10^{-7}$ and $f(-2.618\,033\,95) = -8.7 \times 10^{-8}$, we can say that to seven decimals $x = -2.618\,034\,0$.



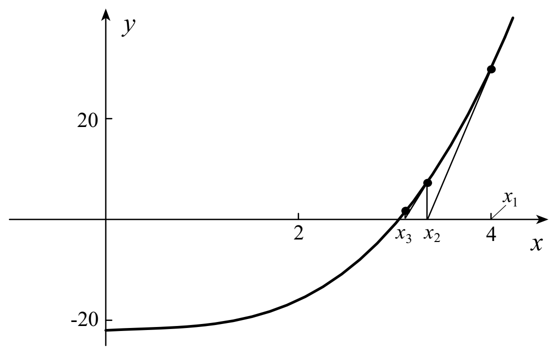

**36.** The figure indicates that with initial approximation $x_1 = 4$, the sequence defined by Newton's iterative procedure has a limit near 3. Iteration of

$$x_1 = 4, \quad x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}$$

leads to

$$x_2 = 3.268, \qquad x_3 = 3.060\,9,$$
$$x_4 = 3.044\,8, \qquad x_5 = 3.044\,723\,15,$$
$$x_6 = 3.044\,723\,15.$$

Since $f(3.044\,723\,05) = -2.2 \times 10^{-6}$ and $f(3.044\,723\,15) = 3.5 \times 10^{-8}$, we can say that to seven decimals $x = 3.044\,723\,1$.



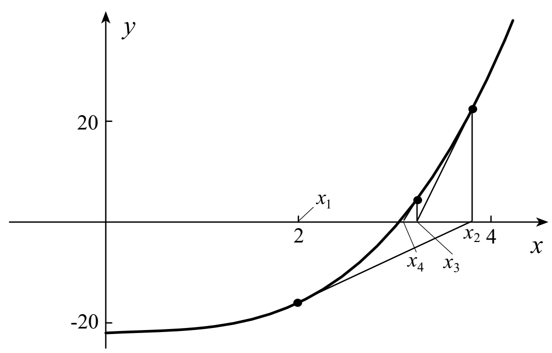

**37.** The figure indicates that with initial approximation $x_1 = 2$, the sequence defined by Newton's iterative procedure has a limit near 3. Iteration of

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}$$

leads to

$$x_2 = 3.778, \qquad x_3 = 3.187,$$
$$x_4 = 3.0515, \qquad x_5 = 3.044\,740,$$
$$x_6 = 3.044\,723\,15, \qquad x_7 = 3.044\,723\,15.$$

Since $f(3.044\,723\,05) = -2.2 \times 10^{-6}$ and $f(3.044\,723\,15) = 3.5 \times 10^{-8}$, we can say that to seven decimals $x = 3.044\,723\,1$.



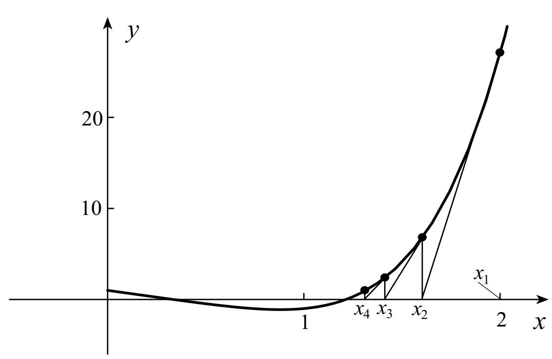

**38.** The figure indicates that with initial aproximation $x_1 = 2$, the sequence defined by Newton's iterative procedure has a limit near 1. Iteration of

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$x_2 = 1.649, \qquad x_3 = 1.406,$$
$$x_4 = 1.268, \qquad x_5 = 1.220,$$
$$x_6 = 1.215, \qquad x_7 = 1.214\,65,$$
$$x_8 = 1.214\,648\,04, \qquad x_9 = 1.214\,648\,04.$$

Since $f(1.214\,647\,95) = -7.3 \times 10^{-7}$ and $f(1.214\,648\,05) = 5.8 \times 10^{-8}$, we can say that to seven decimals $x = 1.214\,648\,0$.



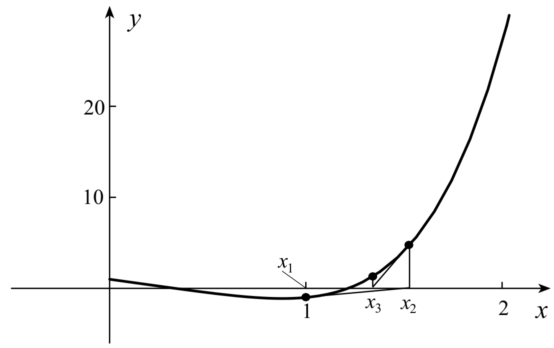
**39.** The figure indicates that the sequence defined by Newton's iterative procedure has a limit. Iteration of

$$x_1 = 1, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$\begin{aligned} x_2 &= 1.5, & x_3 &= 1.317, \\ x_4 &= 1.233, & x_5 &= 1.2154, \\ x_6 &= 1.214\,649, & x_7 &= 1.214\,648\,04, \\ x_8 &= 1.214\,648\,04. \end{aligned}$$

Since $f(1.214\,647\,95) = -7.3 \times 10^{-7}$ and $f(1.214\,648\,05) = 5.8 \times 10^{-8}$, we can say that to seven decimals $x = 1.214\,648\,0$.



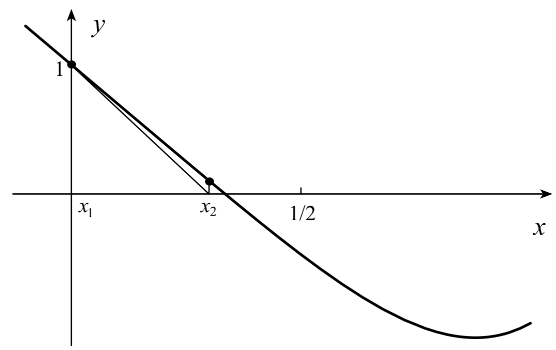

**40.** The figure indicates that with initial approximation $x_1 = 0$, the sequence defined by Newton's iterative procedure has a limit near 0.3. Iteration of

$$x_1 = 0, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$\begin{aligned} x_2 &= 1/3, & x_3 &= 0.334\,7, \\ x_4 &= 0.334\,734\,14, & x_5 &= 0.334\,734\,14. \end{aligned}$$

Since $f(0.334\,734\,05) = 2.7 \times 10^{-7}$ and $f(0.334\,734\,15) = -2.4 \times 10^{-8}$, we can say that to seven decimals $x = 0.334\,734\,1$.



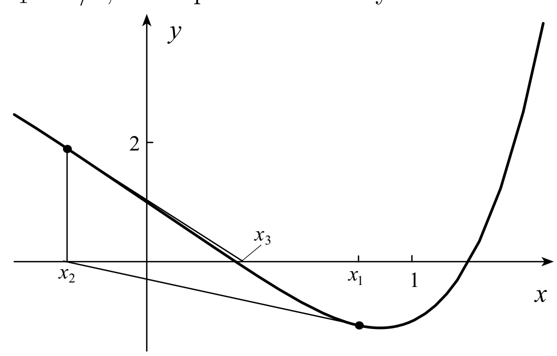

**41.** The figure indicates that with initial approximation $x_1 = 4/5$, the sequence defined by Newton's iterative procedure has a limit near 0.3. Iteration of

$$x_1 = 4/5, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$\begin{aligned} x_2 &= -0.326, & x_3 &= 0.345, \\ x_4 &= 0.334\,72, & x_5 &= 0.334\,734\,14, \\ x_6 &= 0.334\,734\,14. \end{aligned}$$

Since $f(0.334\,734\,05) = 2.7 \times 10^{-7}$ and $f(0.334\,734\,15) = -2.4 \times 10^{-8}$, we can say that to seven decimals $x = 0.334\,734\,1$.



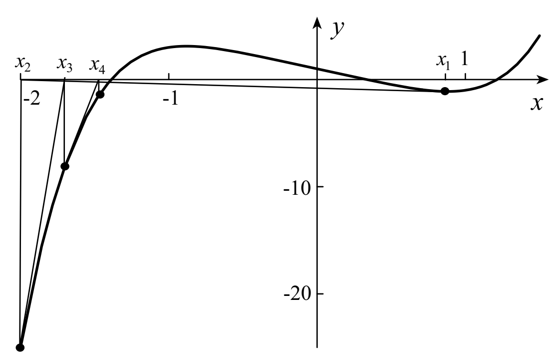

**42.** The figure indicates that with initial approximation $x_1 = 0.85$, the sequence defined by Newton's iterative procedure has a limit near $-1.5$. Iteration of

$$x_1 = 0.85, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$\begin{aligned} x_2 &= -1.987, & x_3 &= -1.667, \\ x_4 &= -1.474, & x_5 &= -1.399, \\ x_6 &= -1.389, & x_7 &= -1.388\,792\,06, \\ x_8 &= -1.388\,791\,98, & x_9 &= -1.388\,791\,98. \end{aligned}$$

Since $f(-1.388\,791\,95) = 5.4 \times 10^{-7}$ and $f(-1.388\,792\,05) = -1.0 \times 10^{-6}$, we can say that to seven decimals $x = -1.388\,792\,0$.



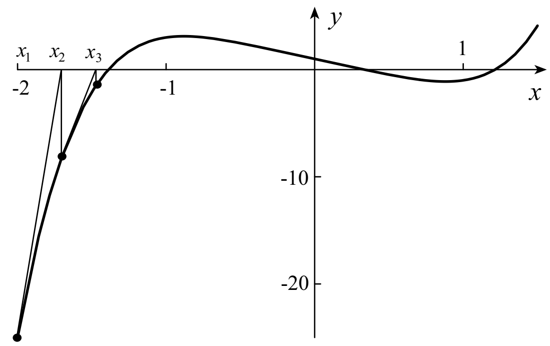
**43.** The figure indicates that with initial approximation $x_1 = -2$, the sequence defined by Newton's iterative procedure has a limit near $-1.5$. Iteration of

$$x_1 = -2, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$x_2 = -1.675, \qquad x_3 = -1.478,$$
$$x_4 = -1.4004, \qquad x_5 = -1.3890,$$
$$x_6 = -1.388\,792, \qquad x_7 = -1.388\,791\,98,$$
$$x_8 = -1.388\,791\,98.$$

Since $f(-1.388\,791\,95) = 5.4 \times 10^{-7}$ and $f(-1.388\,792\,05) = -1.0 \times 10^{-6}$, we can say that to seven decimals $x = -1.388\,792\,0$.




**44.** Iteration of $x_1 = 2, \quad x_{n+1} = 2 + \dfrac{1}{x_n}$ gives

$$x_2 = 2.5, \quad x_3 = 2.4, \quad x_4 = 2.416\,67, \quad x_5 = 2.413\,79, \quad x_6 = 2.414\,29,$$
$$x_7 = 2.414\,20, \quad x_8 = 2.414\,22, \quad x_9 = 2.414\,21, \quad x_{10} = 2.414\,21.$$

Since $f(2.414\,15) = -1.8 \times 10^{-4}$ and $f(2.414\,25) = 1.0 \times 10^{-4}$, it follows that to 4 decimals, $x = 2.4142$.

**45.** Iteration of $x_1 = -1, \quad x_{n+1} = -\dfrac{1}{6}(x_n^3 + 3)$ gives

$$x_2 = -1/3, \quad x_3 = -0.4938, \quad x_4 = -0.4799, \quad x_5 = -0.4816, \quad x_6 = -0.481\,38, \quad x_7 = -0.481\,41.$$

Since $f(-0.481\,35) = 3.7 \times 10^{-4}$ and $f(-0.481\,45) = -3.0 \times 10^{-4}$, it follows that to 4 decimals, $x = -0.4814$.

**46.** Iteration of $x_1 = 0, \quad x_{n+1} = \dfrac{1}{120}(x_n^4 + 20)$ gives

$$x_2 = 1/6, \quad x_3 = 0.166\,67, \quad x_4 = 0.166\,67.$$

Since $f(0.166\,65) = 2.8 \times 10^{-3}$ and $f(0.166\,75) = -9.2 \times 10^{-3}$, it follows that to 4 decimals, $x = 0.1667$.

**47.** Iteration of $x_1 = 3, \quad x_{n+1} = \dfrac{2x_n^2 + 3x_n - 1}{x_n^2}$ gives

$$x_2 = 2.889, \quad x_3 = 2.9186, \quad x_4 = 2.910\,49, \quad x_5 = 2.912\,70,$$
$$x_6 = 2.912\,10, \quad x_7 = 2.912\,26, \quad x_8 = 2.912\,22.$$

Since $f(2.912\,15) = -8.5 \times 10^{-4}$ and $f(2.912\,25) = 2.2 \times 10^{-4}$, the root is $x = 2.9122$ to 4 decimal places.

**48.** Iteration of $x_1 = 0, \quad x_{n+1} = \dfrac{1}{2}(1 + x_n^2)^{1/3}$ gives

$$x_2 = 1/2, \quad x_3 = 0.5386, \quad x_4 = 0.5443, \quad x_5 = 0.545\,17,$$
$$x_6 = 0.545\,31, \quad x_7 = 0.545\,33, \quad x_8 = 0.545\,33.$$

Since $f(0.545\,25) = -4.9 \times 10^{-4}$ and $f(0.545\,35) = 1.2 \times 10^{-4}$, it follows that to 4 decimals, $x = 0.5453$.

**49.** With $x_1 = 3.5$, and $x_{n+1} = \dfrac{6x_n^2 - 11x_n + 7}{x_n^2}$, iteration gives

$$x_2 = 3.4286, \quad x_3 = 3.3872, \quad x_4 = 3.3626, \quad x_5 = 3.3478, \quad x_6 = 3.3388,$$
$$x_7 = 3.333\,34, \quad x_8 = 3.330\,00, \quad x_9 = 3.327\,96, \quad x_{10} = 3.326\,71, \quad x_{11} = 3.325\,94,$$
$$x_{12} = 3.325\,47, \quad x_{13} = 3.325\,18, \quad x_{14} = 3.325\,00, \quad x_{15} = 3.324\,89, \quad x_{16} = 3.324\,82,$$
$$x_{17} = 3.324\,78, \quad x_{18} = 3.324\,76, \quad x_{19} = 3.324\,74.$$

With $f(x) = x^3 - 6x^2 + 11x - 7$, we calculate that $f(3.324\,65) = -2.9 \times 10^{-4}$ and $f(3.324\,75) = 1.4 \times 10^{-4}$. The root is therefore $x = 3.3247$ to 4 decimals.

**50.** With $x_1 = 0$, and $x_{n+1} = \dfrac{x_n^4 - 3x_n^2 + 1}{3}$, iteration gives

$$\begin{array}{lllll} x_2 = 1/3, & x_3 = 0.226, & x_4 = 0.283, & x_5 = 0.255, & x_6 = 0.270, \\ x_7 = 0.262, & x_8 = 0.266, & x_9 = 0.2642, & x_{10} = 0.2652, & x_{11} = 0.2647, \\ x_{12} = 0.2649, & x_{13} = 0.264\,80, & x_{14} = 0.264\,85, & x_{15} = 0.264\,83, & x_{16} = 0.264\,84. \end{array}$$

With $f(x) = x^4 - 3x^2 - 3x + 1$, we calculate that $f(0.264\,75) = 3.9 \times 10^{-4}$ and $f(0.264\,85) = -6.6 \times 10^{-5}$. The root is therefore $x = 0.2648$ to 4 decimals.

**51.** With $x_1 = 0.5$ and $x_{n+1} = \dfrac{50 + 50x_n^2 - 4x_n^3 - x_n^4}{100}$, iteration gives

$$\begin{array}{lllll} x_2 = 0.6194, & x_3 = 0.6809, & x_4 = 0.7170, & x_5 = 0.7397, & x_6 = 0.7544, \\ x_7 = 0.7641, & x_8 = 0.7707, & x_9 = 0.7751, & x_{10} = 0.7782, & x_{11} = 0.7803, \\ x_{12} = 0.7817, & x_{13} = 0.7827, & x_{14} = 0.7834, & x_{15} = 0.7839, & x_{16} = 0.7842, \\ x_{17} = 0.7844, & x_{18} = 0.7846, & x_{19} = 0.7847, & x_{20} = 0.7848, & x_{21} = 0.784\,83, \\ x_{22} = 0.784\,85, & x_{23} = 0.784\,86. & & & \end{array}$$

With $f(x) = x^4 + 4x^3 - 50x^2 + 100x - 50$, we calculate that $f(0.784\,85) = -1.2 \times 10^{-3}$ and $f(0.784\,95) = 1.9 \times 10^{-3}$. Thus to 4 decimals, $x = 0.7849$.

**52.** With $x_1 = 0.75$, and $x_{n+1} = \sqrt{1 - \sin^2 x_n} = \sqrt{\cos^2 x_n} = \cos x_n$, iteration gives

$$\begin{array}{lllll} x_2 = 0.732, & x_3 = 0.744, & x_4 = 0.736, & x_5 = 0.741, & x_6 = 0.738, \\ x_7 = 0.740, & x_8 = 0.7385, & x_9 = 0.7395. & & \end{array}$$

With $f(x) = \sin^2 x - 1 + x^2$, we calculate that $f(0.739\,05) = -8.7 \times 10^{-5}$ and $f(0.739\,15) = 1.6 \times 10^{-4}$. The root is therefore $x = 0.7391$ to 4 decimals.

**53.** By cross-multiplying, $(1 + x^4)\sec x = 2$, and therefore the equation can be rearranged into the form $x = (2\cos x - 1)^{1/4}$. With $x_1 = 0.5$ and $x_{n+1} = (2\cos x_n - 1)^{1/4}$, iteration gives

$$\begin{array}{lllll} x_2 = 0.932, & x_3 = 0.662, & x_4 = 0.872, & x_5 = 0.732, & x_6 = 0.836, \\ x_7 = 0.764, & x_8 = 0.816, & x_9 = 0.780, & x_{10} = 0.806, & x_{11} = 0.788, \\ x_{12} = 0.800, & x_{13} = 0.792, & x_{14} = 0.798, & x_{15} = 0.793, & x_{16} = 0.797, \\ x_{17} = 0.7941, & x_{18} = 0.7962, & x_{19} = 0.7947, & x_{20} = 0.7958, & x_{21} = 0.7950, \\ x_{22} = 0.7956, & x_{23} = 0.7951, & x_{24} = 0.7955, & x_{25} = 0.7952, & x_{26} = 0.7954, \\ x_{27} = 0.7953. & & & & \end{array}$$

With $f(x) = \sec x - 2(1 + x^4)^{-1}$, we calculate that $f(0.795\,25) = -2.6 \times 10^{-4}$ and $f(0.795\,35) = 9.2 \times 10^{-5}$. To 4 decimals then, $x = 0.7953$.

**54.** With $x_1 = 0.5$, and $x_{n+1} = \dfrac{e^{x_n} + e^{-x_n}}{10}$, iteration gives

$$x_2 = 0.226, \quad x_3 = 0.205, \quad x_4 = 0.2042, \quad x_5 = 0.204\,18, \quad x_6 = 0.204\,18.$$

With $f(x) = e^x + e^{-x} - 10x$, we calculate that $f(0.204\,15) = 3.2 \times 10^{-4}$ and $f(0.204\,25) = -6.4 \times 10^{-4}$. The root is therefore $x = 0.2042$ to 4 decimals.

**55.** (a) Iteration of $x_1 = 1$, $x_{n+1} = x_n - \dfrac{x_n^4 - 15x_n + 2}{4x_n^3 - 15}$ gives

$$x_2 = -0.09, \quad x_3 = 0.1333, \quad x_4 = 0.133\,354\,4, \quad x_5 = 0.133\,354\,4.$$

With $f(x) = x^4 - 15x + 2$, we calculate that $f(0.133\,353\,5) = 1.4 \times 10^{-5}$ and $f(0.133\,354\,5) = -1.2 \times 10^{-6}$. The root is therefore $x = 0.133\,354$ to 6 decimals.
(b) Iteration gives

$$x_2 = 0.2, \quad x_3 = 0.133\,44, \quad x_4 = 0.133\,354\,47, \quad x_5 = 0.133\,354\,42.$$

This leads to the same root as in part (a).
(c) Iteration of the sequence in part (a) beginning with $x_1 = 2.5$ gives

$$x_2 = 2.425, \quad x_3 = 2.4201, \quad x_3 = 2.420\,061\,9, \quad x_4 = 2.420\,061\,9.$$

Since $f(2.420\,061\,5) = -1.5 \times 10^{-5}$ and $f(2.420\,062\,5) = 2.7 \times 10^{-5}$, the root is $x = 2.420\,062$.
(d) Iteration beginning with $x_1 = 2$ gives

$$x_2 = 1.2, \quad x_3 = 0.271\,6, \quad x_4 = 0.133\,696, \quad x_5 = 0.133\,355.$$

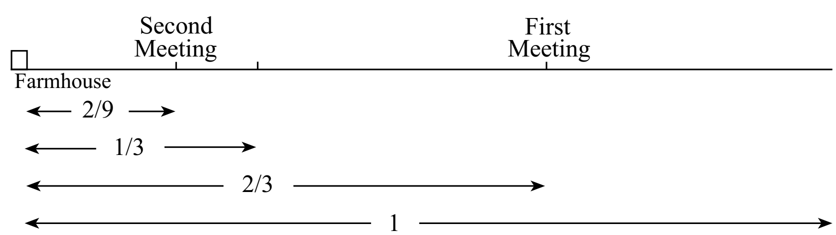
The sequence is converging to the root in part (a). Beginning with $x_1 = 3$, we obtain $x_2 = 5.5$ and $x_3 = 61.1$. The sequence is diverging.

**56.** (a) $d_1 = 2(0.99)(20) = 40(0.99)$ m
$d_2 = 2(0.99)[(0.99)(20)] = 40(0.99)^2$ m
$d_3 = 2(0.99)[(20)(0.99)^2] = 40(0.99)^3$ m
The pattern emerging is $d_n = 40(0.99)^n$ m.

(b) When an object falls from rest under gravity, the distance that it falls as a function of time $t$ is given by $d = 4.905t^2$. Consequently, the time to fall from peak height between $n^{\text{th}}$ and $(n+1)^{\text{th}}$ bounces is given by $d_n/2 = 4.905t^2$. When this equation is solved for $t$, the result is $t = \sqrt{d_n/9.81}$, and therefore

$$t_n = 2\sqrt{d_n/9.81} = 2\sqrt{40(0.99)^n/9.81} = \frac{4}{\sqrt{0.981}}(0.99)^{n/2} \text{ s}.$$

**57.** The dog reaches the farmer for the first time 2/3 km from the farmhouse. When the dog returns to the farmhouse (travelling 2/3 km), the farmer moves to a distance 1/3 km from the farmhouse. The dog then runs $(2/3)(1/3) = 2/9$ km in reaching the farmer for the second time. Thus, $d_1 = 2/3 + 2/9 = 8/9$ km. When the dog returns to the farmhouse for the second time, the farmer moves to a distance 1/9 km from the farmhouse. The dog then runs $(2/3)(1/9) = 2/27$ km in reaching the farmer for the third time. Thus, $d_2 = 2/9 + 2/27 = 8/27$ km. The pattern emerging is $d_n = 8/3^{n+1}$ km.




**58.** Since each of the 12 straight line segments in the middle figure has length $P/9$,

$$P_1 = \frac{12P}{9} = \frac{4P}{3}.$$

Since each of the 48 straight line segments in the right figure has length $P/27$,

$$P_2 = \frac{48P}{27} = \frac{4^2P}{3^2}.$$

The next perimeter is $P_3 = 4(48)\dfrac{P}{81} = \dfrac{4^3P}{3^3}$. The pattern emerging is $P_n = \dfrac{4^nP}{3^n}$. The limit of $P_n$ as $n \to \infty$ does not exist.

**59.** (a) Since $y(3) = 11.8$ and $y(4) = -3.0$, the solution is between 3 and 4. To find it more accurately we use

$$t_1 = 3.8, \qquad t_{n+1} = t_n - \frac{1181(1 - e^{-t_n/10}) - 98.1t_n}{118.1e^{-t_n/10} - 98.1}.$$

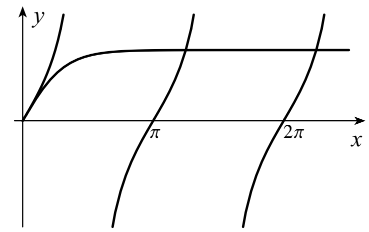
Iteration gives $t_2 = 3.833\,4$ and $t_3 = 3.833\,2$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.

(b) If air resistance is ignored, the acceleration of the stone is $a = dv/dt = -9.81$. Antidifferentiation gives $v(t) = -9.81t + C$. Since $v(0) = 20$, it follows that $C = 20$, and $v(t) = dy/dt = -9.81t + 20$. Antidifferentiation now gives $y(t) = -4.905t^2 + 20t + D$. Since $y(0) = 0$, we find that $D = 0$, and the height of the stone is $y(t) = -4.905t^2 + 20t$. When we set $0 = y = -4.905t^2 + 20t$, the positive solution is 4.08 s.

**60.** The figure shows graphs of $y = \tan x$ and $y = (e^x - e^{-x})/(e^x + e^{-x}) = \tanh x$ for $x \geq 0$. They intersect at $x = 0$ and values near 4 and 7. We use Newton's iterative procedure

$$x_{n+1} = x_n - \frac{\tan x_n - \tanh x_n}{\sec^2 x_n - \operatorname{sech}^2 x_n}$$

with $x_1 = 4$ to locate the smaller root.




Iteration gives $x_2 = 3.932\,25$, $x_3 = 3.926\,63$, $x_4 = 3.926\,60$, $x_5 = 3.926\,60$. When we divide this by $20\pi$, the result is 0.0625. A similar procedure gives the next natural frequency 0.1125.

**61.** Since the area of an equilateral triangle with sides of length $l$ is $\sqrt{3}l^2/4$, the area of the first triangle in Exercise 58 is $\dfrac{\sqrt{3}}{4}\left(\dfrac{P}{3}\right)^2 = \dfrac{\sqrt{3}P^2}{36}$. The middle figure adds three triangles each of area $\sqrt{3}(P/9)^2/4$ to the area in the first figure, and therefore

$$A_1 = \frac{\sqrt{3}P^2}{36} + \frac{3\sqrt{3}}{4}\left(\frac{P^2}{81}\right) = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36}.$$

The right figure adds twelve triangles each of area $\sqrt{3}(P/27)^2/4$ to the middle figure, and therefore

$$A_2 = A_1 + \frac{12\sqrt{3}}{4}\left(\frac{P}{27}\right)^2 = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36} + \frac{4\sqrt{3}P^2}{3^3 \cdot 36}.$$

The next figure in the sequence would add 48 triangles each of area $\sqrt{3}(P/81)^2/4$, and therefore

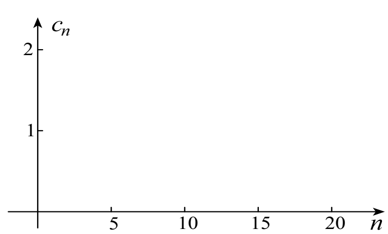
$$A_3 = A_2 + \frac{48\sqrt{3}}{4}\left(\frac{P}{81}\right)^2 = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36} + \frac{4\sqrt{3}P^2}{3^3 \cdot 36} + \frac{4^2\sqrt{3}P^2}{3^5 \cdot 36}.$$

The pattern emerging is $\quad A_n = \dfrac{\sqrt{3}P^2}{36}\left(1 + \dfrac{1}{3} + \dfrac{4}{3^3} + \dfrac{4^2}{3^5} + \cdots + \dfrac{4^{n-1}}{3^{2n-1}}\right).$

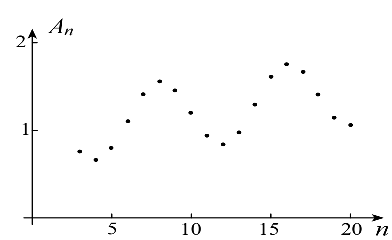
**62.** The next two terms are 1113213211, 31131211131221. Reason as follows: The second term is 11 because there is one 1 in the first term; the third term is 21 because the second term has two 1's; the fourth term is 1211 because the third term has one 2 followed by one 1; the fifth term is 111221 because the fourth term is one 1, followed by one 2, followed by two 1's; etc.

**63.** Plots of the sequences are shown below.

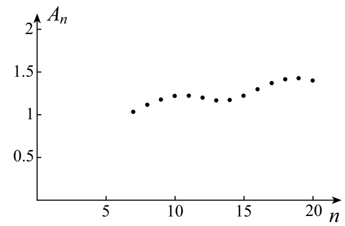
(a)




(b)



**64.** The plot of the seven-point averager is shown to the right.




**65.** An explicit formula for this FIR is $F_n = \dfrac{n}{n+1} + 2\left(\dfrac{n-1}{n}\right) - \left(\dfrac{n-2}{n-1}\right)$. When we substitute $n = 3, \ldots, 12$, we obtain the first 10 terms,

$$\frac{19}{12}, \frac{49}{30}, \frac{101}{60}, \frac{181}{105}, \frac{295}{168}, \frac{449}{252}, \frac{649}{360}, \frac{901}{495}, \frac{1211}{660}, \frac{1585}{858}.$$

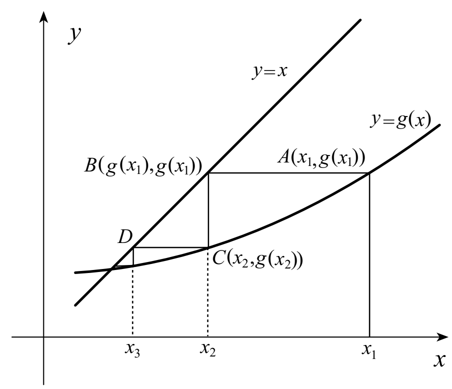
**66.** An explicit formula for this FIR is

$$F_n = \frac{1}{n^2}\sin\left(\frac{n}{3}\right) - \frac{2}{(n-1)^2}\sin\left(\frac{n-1}{3}\right) + \frac{3}{(n-2)^2}\sin\left(\frac{n-2}{3}\right) - \frac{4}{(n-3)^2}\sin\left(\frac{n-3}{3}\right).$$

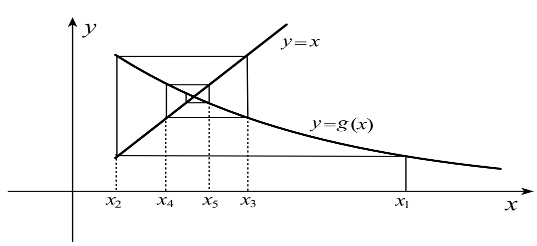
When we substitute $n = 4, \ldots, 13$, we obtain the first 10 terms,

$$-0.9712, -0.4196, -0.2461, -0.1593, -0.1059, -0.0693, -0.0430, -0.0237, -0.0096, 0.0002.$$

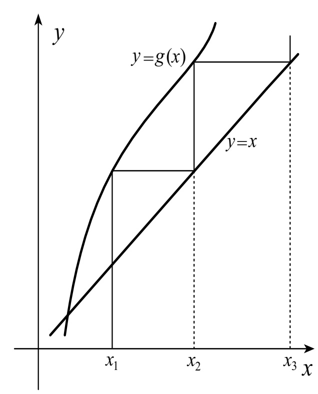
**67.** (a) The height of the curve $y = g(x)$ at the point $A$ with $x$-coordinate $x_1$ is $y = g(x_1)$. If we proceed horizontally to the line $y = x$, the coordinates of the point $B$ on the line are $\big(g(x_1), g(x_1)\big)$. But the second term in the sequence established by the method of successive substitutions is $x_2 = g(x_1)$. Hence the $x$-coordinate of $B$ is $x_2$. The height of the curve $y = g(x)$ at $C$ is $y = g(x_2)$. The point $D$ has coordinates $\big(g(x_2), g(x_2)\big)$, and hence, the $x$-coordinate of $D$ is $x_3 = g(x_2)$. Continuation leads to the interpretation of the $\{x_n\}$ as shown in the figure.




(b)

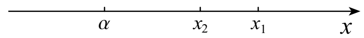



(c)




(d) It appears that the slope of $y = g(x)$ near the required root dictates whether the sequence converges. For slopes near zero (figures in (a) and (b)), the sequence converges, but for large slopes (figure in (c)), the sequence diverges.

(e)




If we apply the mean value theorem (Theorem '`mean value theorem`') to $g(x)$ on the interval between $\alpha$ and $x_1$,

$$g(x_1) = g(\alpha) + g'(c)(x_1 - \alpha)$$

where $c$ is between $\alpha$ and $x_1$. Since $\alpha = g(\alpha)$, $x_2 = g(x_1)$, and $|g'(c)| \le a$, we may write that

$$x_2 = \alpha + g'(c)(x_1 - \alpha) \quad \Longrightarrow \quad |x_2 - \alpha| = |g'(c)||x_1 - \alpha| \le a|x_1 - \alpha|.$$

What this means is that $x_2$ is closer to $\alpha$ than $x_1$. If we repeat this procedure for $x_3 = g(x_2)$ on the interval between $\alpha$ and $x_2$, we obtain

$$|x_3 - \alpha| \le a|x_2 - \alpha| \le a^2|x_1 - \alpha|.$$

Continuation of this process gives $|x_n - \alpha| \le a^{n-1}|x_1 - \alpha|$. It now follows that

$$\lim_{n\to\infty} |x_n - \alpha| \le \lim_{n\to\infty} a^{n-1}|x_1 - \alpha| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} x_n = \alpha.$$

**68.** Newton's iterative procedure defines the sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. If we define $F(x) = x - f(x)/f'(x)$, then $x_{n+1} = F(x_n)$. According to part (e) of Exercise 67, a sequence of this type converges to a root $x = \alpha$ of $x = F(x)$ if on the interval $|x - \alpha| \le |x_1 - \alpha|$ we have $|F'(x)| \le a < 1$. Since $F'(x) = 1 - [(f')^2 - ff'']/(f')^2$, we will have convergence if $1 > a \ge \left|1 - \dfrac{(f')^2 - ff''}{(f')^2}\right| = \left|\dfrac{ff''}{(f')^2}\right|$. Thus, Newton's sequence converges to $\alpha$ if on $|x-\alpha| \le |x_1-\alpha|$, $|ff''/(f')^2| \le a < 1$. In other words, if it is possible to choose $x_1$ close enough to $\alpha$ to guarantee $|ff''/(f')^2| \le a < 1$, on the interval $|x - \alpha| \le |x_1 - \alpha|$, then Newton's sequence converges to $\alpha$. To show that this is always possible, we let $M$ be the maximum value of $|f''|$ on the open interval containing $\alpha$ in which $f''(x)$ is known to exist. Because $f'(\alpha) \ne 0$, there exists an open interval $I$ containing $\alpha$ in which $f'(x) \ne 0$ (by continuity of $f'(x)$). Let $m$ be the minimum value of $|f'(x)|$ on $I$. Since $f(x)$ is continuous at $x = \alpha$, where $f(\alpha) = 0$, there exists an open interval $|x - \alpha| < \delta$ contained in $I$ which $|f(x)| < am^2/M$ for any $a$ such that $0 < a < 1$. Consequently, for $|x - \alpha| < \delta$,

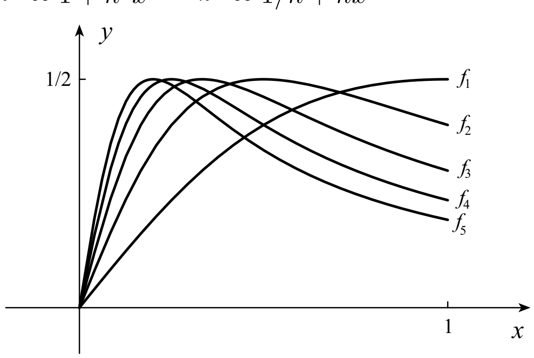
$$\left|\frac{ff''}{(f')^2}\right| < \frac{am^2}{M}\frac{M}{m^2} = a < 1.$$

Thus, if $|x_1 - \alpha| = \delta$, we may say that for all $x$ in $|x - a| < |x_1 - \alpha|$, $|ff''/(f')^2| < a < 1$, and Newton's iterative sequence converges to $\alpha$.

## EXERCISES 10.2

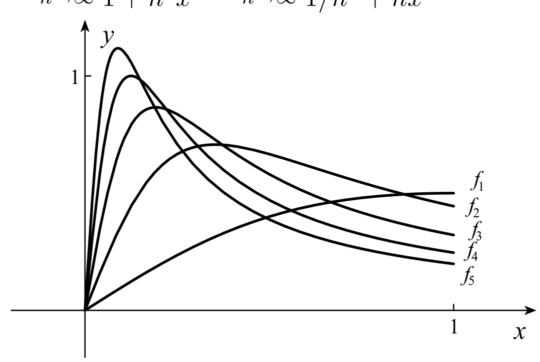
**1.** The limit function is $f(x) = 0$, since for each $x$ in $0 \le x \le 1$,

$$\lim_{n\to\infty} \frac{nx}{1 + n^2x^2} = \lim_{n\to\infty} \frac{x}{1/n + nx^2} = 0.$$

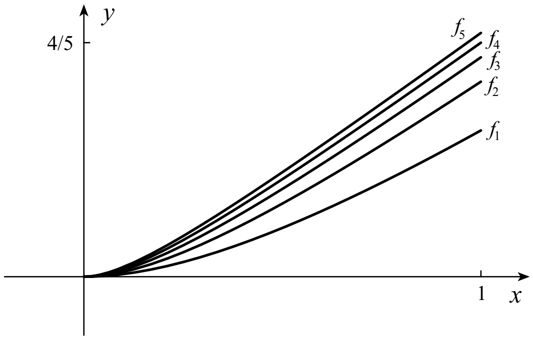



**2.** The limit function is $f(x) = 0$, since for each $x$ in $0 \le x \le 1$,

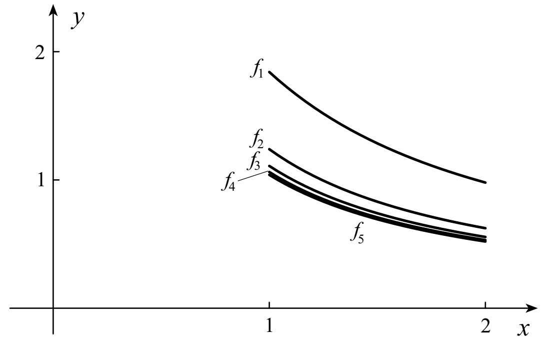
$$\lim_{n\to\infty} \frac{n^2x}{1 + n^3x^2} = \lim_{n\to\infty} \frac{x}{1/n^2 + nx^2} = 0.$$



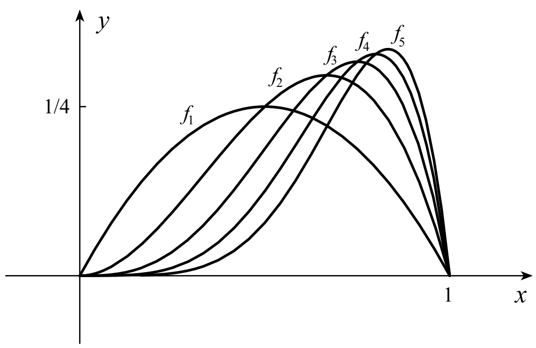
**3.** The limit function is $f(x) = x$, since for each $x$ in $0 \le x \le 1$,
$$\lim_{n\to\infty} \frac{nx^2}{1+nx} = \lim_{n\to\infty} \frac{x^2}{1/n + x} = x.$$



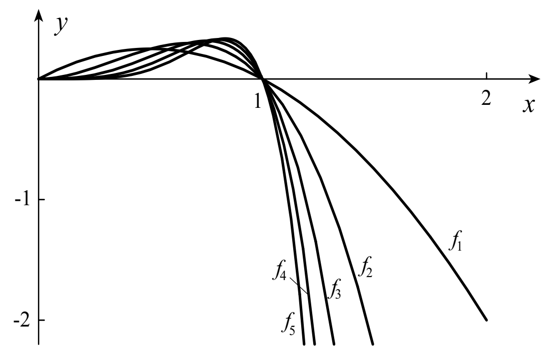

**4.** The limit function is $f(x) = 1/x$.



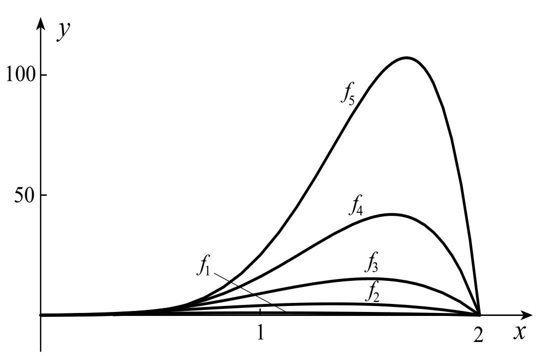

**5.** Since $f_n(0) = f_n(1) = 0$, the limit function $f(x)$ has values $f(0) = f(1) = 0$. For fixed $x$ in $0 < x < 1$,
$$f(x) = \lim_{n\to\infty} nx^n(1-x) = \lim_{n\to\infty} \frac{n(1-x)}{x^{-n}}$$
$$= \lim_{n\to\infty} \frac{1-x}{-x^{-n}\ln x} = \lim_{n\to\infty} \frac{x^n(x-1)}{\ln x} = 0.$$



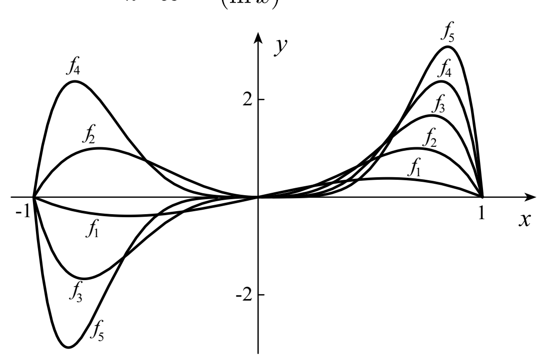

**6.** There is no limit function.



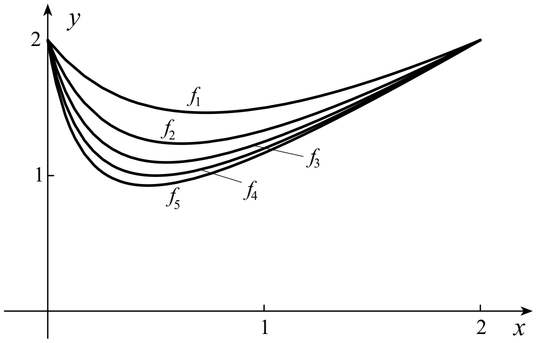

**7.** There is no limit function.



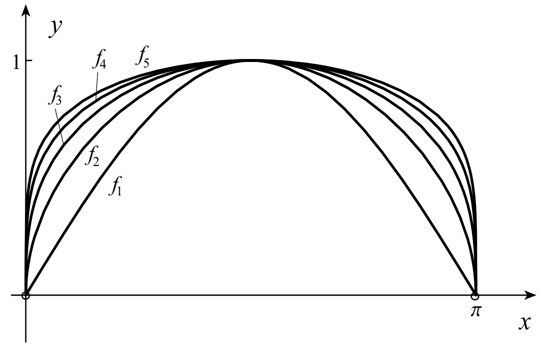

**8.** Since $f_n(0) = f_n(1) = 0$, the limit function $f(x)$ has values $f(0) = f(1) = 0$. For fixed $x$ in $0 < x < 1$,
$$f(x) = \lim_{n\to\infty} n^2x^n(1-x^2) = \lim_{n\to\infty} \frac{n^2(1-x^2)}{x^{-n}}$$
$$= \lim_{n\to\infty} \frac{2n(1-x^2)}{-x^{-n}\ln x} = \lim_{n\to\infty} \frac{2(1-x^2)}{x^{-n}(\ln x)^2}$$
$$= \lim_{n\to\infty} \frac{2(1-x^2)x^n}{(\ln x)^2} = 0.$$



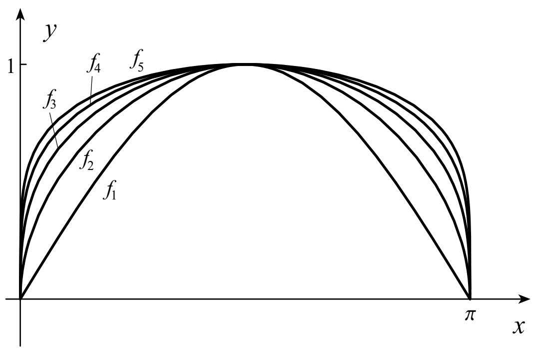
**9.** The limit function $f(x)$ has value 2 at $x=0$, and for all other vaues of $x$,
$$f(x)=\lim_{n\to\infty}\frac{2+nx^2}{1+nx}=\lim_{n\to\infty}\frac{2/n+x^2}{1/n+x}=x.$$



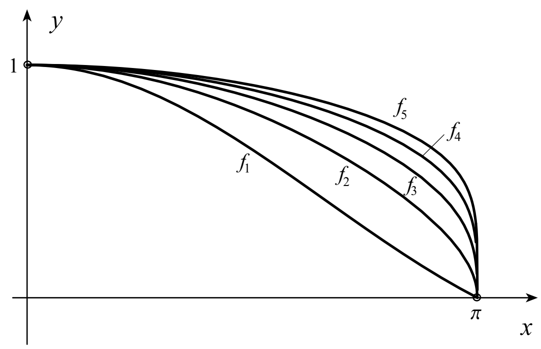

**10.** The limit function is $f(x)=1$.



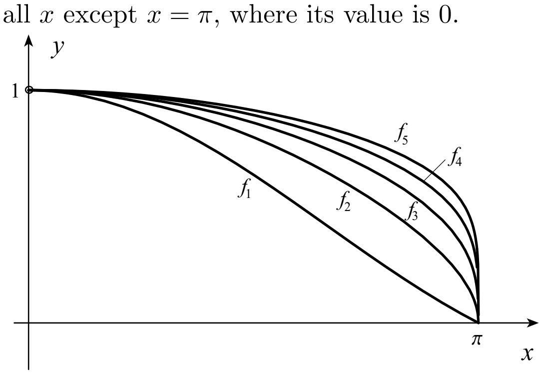

**11.** The limit function $f(x)$ has value 1 for all $x$ except $x=0,\pi$, where its value is 0.



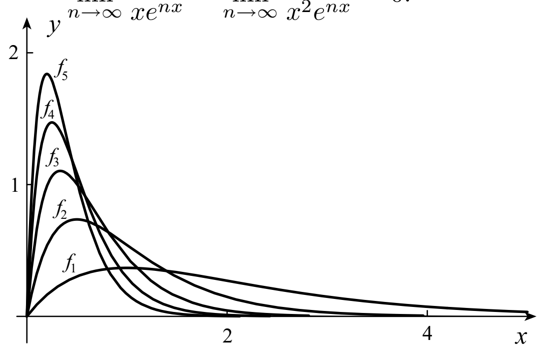

**12.** The limit function is $f(x)=1$.




**13.** The limit function $f(x)$ has value 1 for all $x$ except $x=\pi$, where its value is 0.




**14.** The limit function is $f(x)=\lim_{n\to\infty}\dfrac{n^2x}{e^{nx}}$
$$=\lim_{n\to\infty}\frac{2nx}{xe^{nx}}=\lim_{n\to\infty}\frac{2x}{x^2e^{nx}}=0.$$




**15.** The sequence $\{x^n\}$ converges to 0 for $-1<x<1$, to 1 for $x=1$, and diverges for all other values of $x$. Hence, the sequence $\{(1-x^n)/(1-x)\}$ converges to $1/(1-x)$ for $-1<x<1$ and diverges for all other values of $x$.

## EXERCISES 10.3

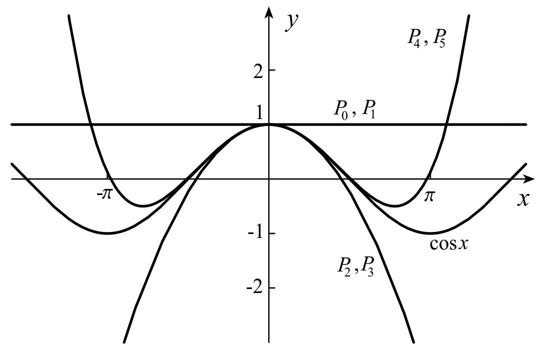
**1.** Since $f(0) = 1$, $f'(0) = -\sin 0 = 0$, $f''(0) = -\cos 0 = -1$, $f'''(0) = \sin 0 = 0$, $f''''(0) = \cos 0 = 1$, etc., Taylor's remainder formula gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \text{term in } x^n + R_n,$$

where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}(\cos x)_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!}$. The $n^{\text{th}}$ derivative of $\cos x$ is $\pm \sin x$ or $\pm \cos x$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}} \cos x_{|x=z_n} \right| \leq 1.$$

Hence, $|R_n| \leq |x|^{n+1}/(n+1)!$. But according to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any



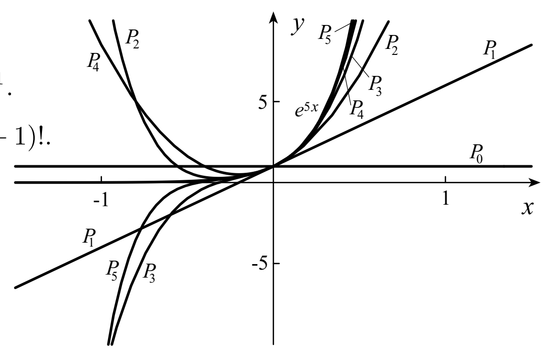

$x$ whatsoever. It follows that $\lim_{n\to\infty} R_n = 0$, and the Maclaurin series for $\cos x$ therefore converges to $\cos x$ for all $x$. We may write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty.$$

**2.** Since $f^{(n)}(x) = 5^n e^{5x}$, Taylor's remainder formula for $e^{5x}$ and $c = 0$ gives

$$e^{5x} = 1 + 5x + \frac{5^2}{2!}x^2 + \frac{5^3}{3!}x^3 + \cdots + \frac{5^n}{n!}x^n + R_n,$$

where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}(e^{5x})_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!} = \dfrac{5^{n+1}e^{5z_n}}{(n+1)!}x^{n+1}$.
If $x < 0$, then $x < z_n < 0$, and $|R_n| < 5^{n+1}|x|^{n+1}/(n+1)!$.
According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any $x$ whatsoever, and therefore $\lim_{n\to\infty} 5^{n+1}|x|^{n+1}/(n+1)! = 0$ also.
Thus, if $x < 0$, $\lim_{n\to\infty} R_n = 0$. If $x > 0$,




then $0 < z_n < x$, and

$$|R_n| < \frac{5^{n+1}e^{5x}}{(n+1)!}|x|^{n+1} = e^{5x}\left(\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\right).$$

But we have just indicated that $\lim_{n\to\infty} 5^{n+1}|x|^{n+1}/(n+1)! = 0$, and therefore $\lim_{n\to\infty} R_n = 0$ for $x > 0$ also. Thus, for any $x$ whatsoever, the sequence $\{R_n\}$ has limit 0, and the Maclaurin series for $e^{5x}$ converges to $e^{5x}$,

$$e^{5x} = \sum_{n=0}^{\infty} \frac{5^n}{n!}x^n, \quad -\infty < x < \infty.$$

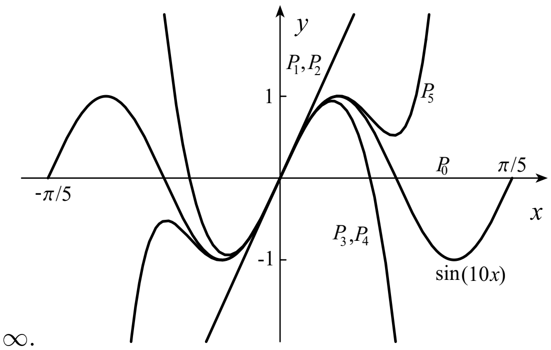
**3.** Since $f(0) = \sin(0) = 0$, $f'(0) = 10\cos 0 = 10$, $f''(0) = -10^2 \sin 0 = 0$, $f'''(0) = -10^3 \cos 0 = -10^3$, $f''''(0) = 10^4 \sin 0 = 0$, etc., Taylor's remainder formula gives

$$\sin(10x) = 10x - \frac{10^3x^3}{3!} + \frac{10^5x^5}{5!} + \cdots + \text{term in } x^n + R_n,$$

where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}[\sin(10x)]_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!}$. The $n^{\text{th}}$ derivative of $\sin(10x)$ is $\pm 10^n \sin(10x)$ or $\pm 10^n \cos(10x)$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}}[\sin(10x)]_{|x=z_n} \right| \le 10^{n+1}.$$

Hence, $|R_n| \le 10^{n+1}|x|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any $x$ whatsoever, and therefore $\lim_{n\to\infty} 10^{n+1}|x|^{n+1}/(n+1)! = 0$ also. It follows that $\lim_{n\to\infty} R_n = 0$, and the Maclaurin series for $\sin(10x)$ therefore converges to $\sin(10x)$ for all $x$.




We may write

$$\sin(10x) = 10x - \frac{10^3x^3}{3!} + \frac{10^5x^5}{5!} + \cdots, \quad -\infty < x < \infty.$$

**4.** Since $f(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, $f'(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, $f''(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$, $f'''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}$, $f''''(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, etc., Taylor's remainder formula gives

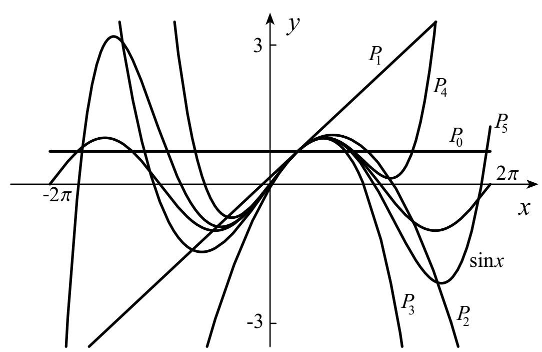
$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x-\pi/4) - \frac{1}{2!\sqrt{2}}(x-\pi/4)^2 - \frac{1}{3!\sqrt{2}}(x-\pi/4)^3 + \ldots + \text{term in } (x-\pi/4)^n + R_n,$$

where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}(\sin x)_{|x=z_n} \dfrac{(x-\pi/4)^{n+1}}{(n+1)!}$.

The $n^{\text{th}}$ derivative of $\sin x$ is $\pm \sin x$ or $\pm \cos x$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}}(\sin x)_{|x=z_n} \right| \le 1.$$

Hence, $|R_n| \le |x-\pi/4|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any $x$ whatsoever, and therefore $\lim_{n\to\infty} |x-\pi/4|^{n+1}/(n+1)! = 0$ also. It follows that $\lim_{n\to\infty} R_n = 0$, and the Taylor series for $\sin x$ about $\pi/4$ therefore converges to $\sin x$ for all $x$.




We may write

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + (x-\pi/4) - \frac{1}{2!}(x-\pi/4)^2 - \frac{1}{3!}(x-\pi/4)^3 + \cdots\right], \quad -\infty < x < \infty.$$

**5.** Since $f^{(n)}(x) = 2^n e^{2x}$, Taylor's remainder formula for $e^{2x}$ and $c = 1$ gives

$$e^{2x} = e^2 + 2e^2(x-1) + \frac{2^2e^2}{2!}(x-1)^2 + \frac{2^3e^2}{3!}(x-1)^3 + \cdots + \frac{2^ne^2}{n!}(x-1)^n + R_n,$$

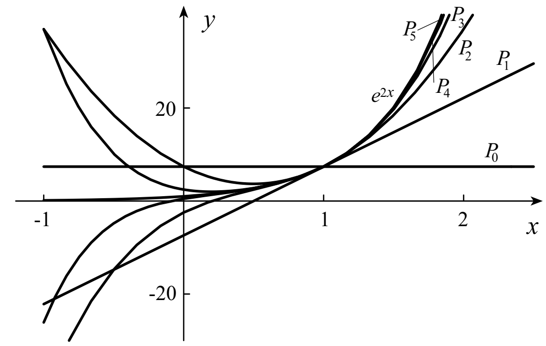
where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}(e^{2x})_{|x=z_n} \dfrac{(x-1)^{n+1}}{(n+1)!} = \dfrac{2^{n+1}e^{2z_n}}{(n+1)!}(x-1)^{n+1}$. If $x < 1$, then $x < z_n < 1$, and $|R_n| < 2^{n+1}e^2|x-1|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any $x$ whatsoever, and therefore $\lim_{n\to\infty} 2^{n+1}e^2|x-1|^{n+1}/(n+1)! = 0$ also.

Thus, if $x < 1$, $\lim_{n\to\infty} R_n = 0$. If $x > 1$, then $1 < z_n < x$, and

$$|R_n| < \frac{2^{n+1}e^{2x}}{(n+1)!}|x-1|^{n+1} = e^{2x}\left[\frac{2^{n+1}|x-1|^{n+1}}{(n+1)!}\right].$$

But we have just indicated that $\lim_{n\to\infty} 2^{n+1}|x-1|^{n+1}/(n+1)! = 0$, and therefore $\lim_{n\to\infty} R_n = 0$ for $x > 1$ also. Thus, for any $x$ whatsoever, the sequence $\{R_n\}$ has limit 0, and the Taylor series for $e^{2x}$ converges to $e^{2x}$,

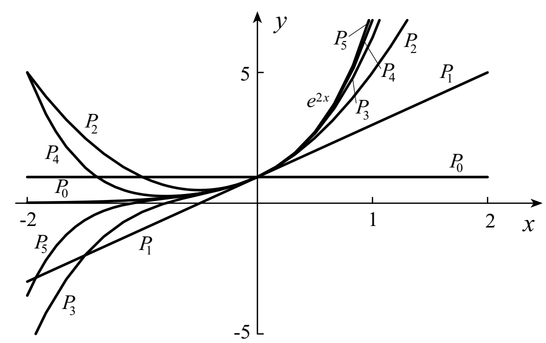
$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n, \quad -\infty < x < \infty.$$




**6.** Since $f^{(n)}(0) = 2^n$, the Maclaurin series for $e^{2x}$ is

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = 1 + 2x + \frac{2^2x^2}{2!} + \cdots.$$

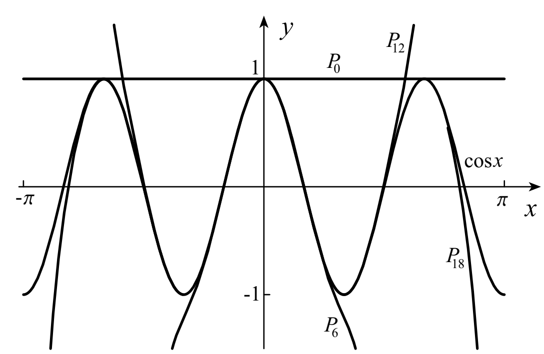
Plots of the polynomials suggest that the series converges to $e^{2x}$ for all $x$.




**7.** Since $f(0) = 1$, $f'(0) = 0$, $f''(0) = -3^2$, $f'''(0) = 0$, $f''''(0) = 3^4$, etc., the Maclaurin series for $\cos 3x$ is

$$1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!}x^{2n}.$$

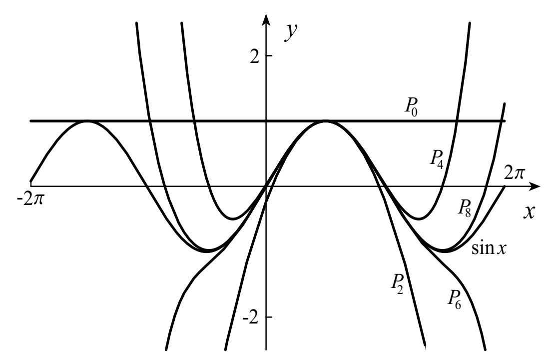
Plots of the polynomials suggest that the series converges to $\cos 3x$ for all $x$.




**8.** Since $f(\pi/2) = 1$, $f'(\pi/2) = 0$, $f''(\pi/2) = -1$, $f'''(\pi/2) = 0$, and $f''''(\pi/2) = 1$, the Taylor series for $\sin x$ about $x = \pi/2$ is

$$1 - \frac{(x-\pi/2)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n}.$$

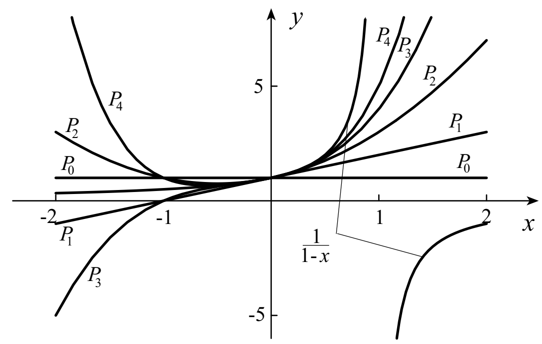
Plots of the polynomials suggest that the series converges to $\sin x$ for all $x$.



**9.** Since $f^{(n)}(0) = n!$, the Maclaurin series for $1/(1-x)$ is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

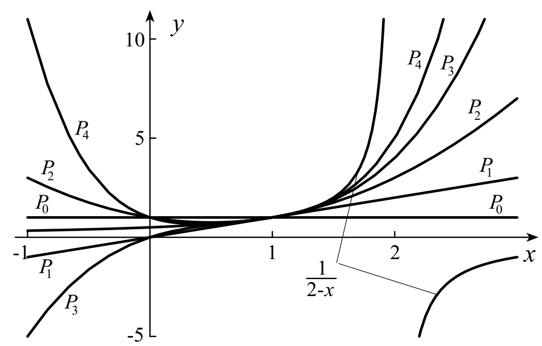
Plots of the polynomials suggest that the series converges to $1/(1-x)$ for $-1 < x < 1$.




**10.** Since $f^{(n)}(1) = n!$, the Taylor series for $1/(2-x)$ about $x = 1$ is

$$\sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots .$$

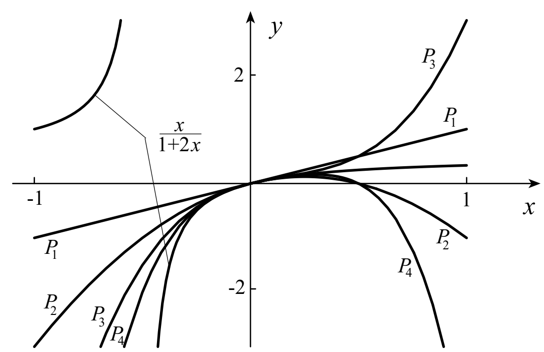
Plots of the polynomials suggest that the series converges to $1/(2-x)$ only for $0 < x < 2$.




**11.** By writing $f(x)$ in the form $1/2 - (1/2)/(1+2x)$ and taking derivatives, we quickly discover that $f^{(n)}(0) = (-1)^{n+1}2^{n-1}n!$ for $n \geq 1$. The Maclaurin series for $f(x)$ is therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1}2^{n-1}x^n = x - 2x^2 + 4x^3 - 8x^4 + \cdots .$$

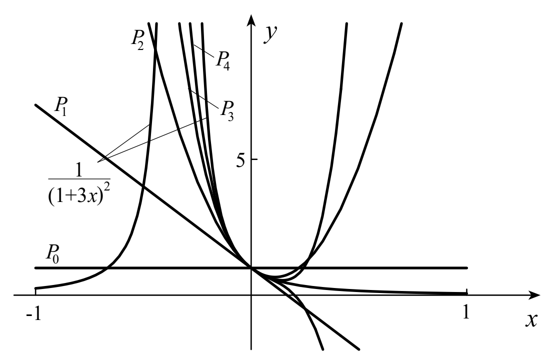
Plots of the polynomials suggest that the series converges to $x/(1+2x)$ only for $-1/2 < x < 1/2$.




**12.** Since $f^{(n)}(0) = (-1)^n 3^n (n+1)!$, the Maclaurin series for $1/(1+3x)^2$ is

$$\sum_{n=0}^{\infty} (-1)^n 3^n (n+1)x^n = 1 - 6x + 3^2(3)x^2 + \cdots .$$

Plots of the polynomials suggest that the series converges to $1/(1+3x)^2$ only for $-1/3 < x < 1/3$.

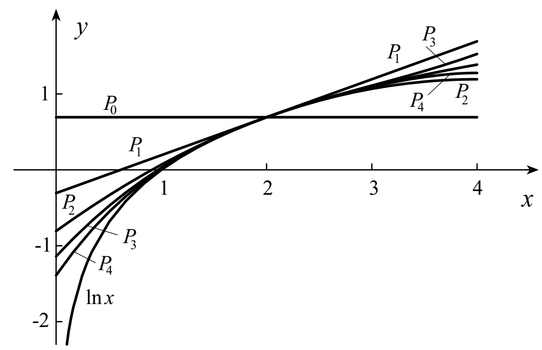


**13.** Since $f^{(n)}(2) = (-1)^{n+1}(n-1)!/2^n$ for $n \geq 1$, the Taylor series for $\ln x$ about $x = 2$ is

$$\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}(x-2)^n$$

$$= \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{2 \cdot 2^2} + \cdots.$$

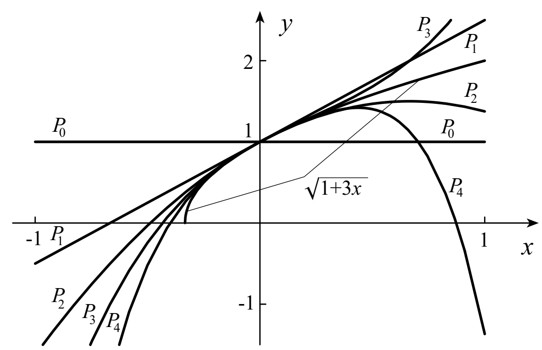
Plots of the polynomials suggest that the series converges to $\ln x$ only for $0 < x < 4$.




**14.** Calculating derivatives of the function leads to the formula $f^{(n)}(0) = \dfrac{(-1)^{n+1}3^n[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{2^n}$ for $n \geq 2$, together with $f(0) = 1$ and $f'(0) = 3/2$. The Maclaurin series for $\sqrt{1+3x}$ is therefore

$$1 + \frac{3x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}3^n[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{2^n\,n!}x^n.$$

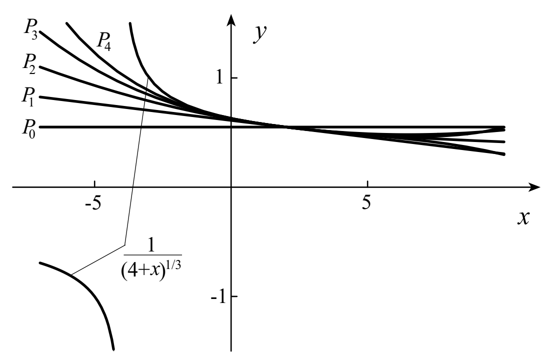
Plots of the polynomials suggest that the series converges to $\sqrt{1+3x}$ only for $-1/3 \leq x \leq 1/3$.




**15.** Calculating derivatives of the function leads to the formula $f^{(n)}(2) = \dfrac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{2^n 3^{2n} 6^{1/3}}$ for $n \geq 1$. The Taylor series for $1/(4+x)^{1/3}$ about $x = 2$ is therefore

$$\frac{1}{6^{1/3}} + \sum_{n=1}^{\infty} \frac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{2^n 3^{2n} 6^{1/3} n!}(x-2)^n.$$

Plots of the polynomials suggest that the series converges to $1/(4+x)^{1/3}$ only for $-4 < x < 8$.




**16.** If $I'$ is the open interval in which $f'(x)$ and $f''(x)$ are continuous, and we apply Taylor's remainder formula to $f(x)$ at $x_0$ in $I'$, we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(z_1)}{2!}(x - x_0)^2 = f(x_0) + \frac{f''(z_1)}{2}(x - x_0)^2,$$

where $z_1$ is between $x_0$ and $x$. Suppose that $f''(x_0) > 0$. Because $f''(x)$ is continuous at $x_0$, there exists an open interval $I$ containing $x_0$ in which $f''(x) > 0$. For any $x$ in this interval, it follows that $f''(z_1) > 0$ also. As a result, for any $x$ in $I$, $f(x) > f(x_0)$, and $f(x)$ must have a relative minimum at $x_0$. A similar discussion shows that when $f''(x_0) < 0$, the function has a relative maximum at $x_0$. If $f''(x_0) = 0$, no conclusion can be reached.

**17.** If $I'$ is the open interval in which $f(x)$ has derivatives of all orders, and we apply Taylor's remainder formula to $f(x)$ at $x_0$ in $I'$, we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(z_n)}{(n+1)!}(x - x_0)^{n+1}$$

$$= f(x_0) + \frac{f^{(n+1)}(z_n)}{(n+1)!}(x - x_0)^{n+1}$$

where $z_n$ is between $x_0$ and $x$.

(i) Consider first the case that $n$ is even, and suppose that $f^{(n+1)}(x_0) > 0$. (A similar proof follows in the case that $f^{(n+1)}(x_0) < 0$.) Because $f^{(n+1)}(x)$ is continuous at $x_0$, there exists an open interval $I$ containing $x_0$ in which $f^{(n+1)}(x) > 0$. For any $x$ in this interval, it follows that $f^{(n+1)}(z_n) > 0$ also. As a result, when $x < x_0$, $f(x) < f(x_0)$, and when $x > x_0$, $f(x) > f(x_0)$. This implies that $x_0$ must yield a horizontal point of inflection.

(ii) Consider now when $n$ is odd and $f^{(n+1)}(x_0) > 0$. In this case, for any $x$ in $I$, $f(x) > f(x_0)$ and $f(x)$ must have a relative minimum at $x_0$.

(iii) When $n$ is odd and $f^{(n+1)}(x_0) < 0$, $f(x) < f(x_0)$ in $I$, and $f(x)$ has a relative maximum at $x_0$.

**18.** (a) This follows from $\displaystyle\int_c^x f'(t)\,dt = \Big\{f(t)\Big\}_c^x = f(x) - f(c)$.

(b) If we set $u = f'(t)$, $du = f''(t)\,dt$, $dv = dt$, and $v = t - x$, then

$$f(x) = f(c) + \Big\{(t-x)f'(t)\Big\}_c^x - \int_c^x (t-x)f''(t)\,dt = f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)\,dt.$$

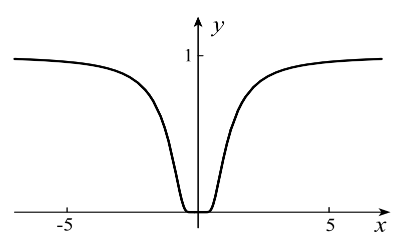
(c) If we now set $u = f''(t)$, $du = f'''(t)\,dt$, $dv = (x-t)\,dt$, and $v = -(1/2)(x-t)^2$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \left\{-\frac{(x-t)^2 f''(t)}{2}\right\}_c^x - \int_c^x -\frac{1}{2}(x-t)^2 f'''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{1}{2}\int_c^x (x-t)^2 f'''(t)\,dt. \end{aligned}$$

(d) One more integration by parts should convince us that the formula is correct. If we set $u = f'''(t)$, $du = f''''(t)\,dt$, $dv = (x-t)^2\,dt$, and $v = -(1/3)(x-t)^3$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{1}{2}\left\{-\frac{(x-t)^3 f'''(t)}{3}\right\}_c^x - \frac{1}{2}\int_c^x -\frac{1}{3}(x-t)^3 f''''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{1}{3!}\int_c^x (x-t)^3 f''''(t)\,dt. \end{aligned}$$

**19.** (a) Limits as $x \to 0^+$ and $x \to \infty$ together with symmetry about the $y$-axis give the graph to the right.




(b) If we can show that

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0,$$

then the limit from the left must also be zero.

Suppose we set $L = \displaystyle\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n}$, and take logarithms,

$$\ln L = -\lim_{x\to 0^+}\left(\frac{1}{x^2} + n\ln x\right) = -\lim_{x\to 0^+}\left(\frac{1 + nx^2\ln x}{x^2}\right).$$

Since $\displaystyle\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}(-x^2/2) = 0$, it follows that $\ln L \to -\infty$ as $x \to 0^+$. Therefore, $L = 0$.

(c) $f'(0) = \displaystyle\lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = 0$, by part (b). Suppose that $k$ is some integer for which $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{h\to 0}\frac{f^{(k)}(h) - f^k(0)}{h} = \lim_{h\to 0}\frac{f^{(k)}(h)}{h}.$$

Now, any number of differentiations of $f(x) = e^{-1/x^2}$ gives rise to terms of the form $Ae^{-1/x^2}/x^n$, where $n$ is a positive integer, and $A$ is a constant. It follows that $f^{(k)}(h)/h$ must consist of terms of the form $Ae^{-1/h^2}/h^n$ which have limit zero as $h \to 0$. Thus, $f^{(k+1)}(0) = 0$, and by mathematical induction, $f^{(n)}(0) = 0$ for all $n \geq 1$.

(d) The Maclaurin series for $f(x)$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = 0 + 0 + 0 + \cdots \ .$$

(e) This series converges to $f(x)$ only at $x = 0$.

**EXERCISES 10.4**

**1.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{1/n}{1/(n+1)}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

**2.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{n^2}{(n+1)^2}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

**3.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{1/(n+1)^3}{1/(n+2)^3}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

**4.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{n^2 3^n}{(n+1)^2 3^{n+1}}\right| = \dfrac{1}{3}$, the open interval of convergence is $-1/3 < x < 1/3$.

**5.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{1/2^n}{1/2^{n+1}}\right| = 2$, the open interval of convergence is $-1 < x < 3$.

**6.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{(-1)^n n^3}{(-1)^{n+1}(n+1)^3}\right| = 1$, the open interval of convergence is $-4 < x < -2$.

**7.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{1/\sqrt{n}}{1/\sqrt{n+1}}\right| = 1$, the open interval of convergence is $-3 < x < -1$.

**8.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{2^n \left(\dfrac{n-1}{n+2}\right)^2}{2^{n+1}\left(\dfrac{n}{n+3}\right)^2}\right| = \dfrac{1}{2}$, the open interval of convergence is $7/2 < x < 9/2$.

**9.** If we set $y = x^2$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}x^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}y^n$. Since $R_y = \lim_{n\to\infty} \left|\dfrac{1/n^2}{1/(n+1)^2}\right| = 1$, it follows that $R_x = \sqrt{R_y} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

**10.** If we set $y = x^3$, then $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n y^n$. Since $R_y = \lim_{n\to\infty} \left|\dfrac{(-1)^n}{(-1)^{n+1}}\right| = 1$, it follows that $R_x = {R_y}^{1/3} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

**11.** Since the radius of convergence is $R = \lim_{n\to\infty} \left|\dfrac{2^n(n-1)/(n+1)}{2^{n+1}n/(n+2)}\right| = 1/2$, the open interval of convergence is $-1/2 < x < 1/2$.

**12.** If we set $y = x^3$, then $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}x^{3n+1} = y^{1/3}\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}y^n$. Since $R_y = \lim_{n\to\infty} \left|\dfrac{1/\sqrt{n+1}}{1/\sqrt{n+2}}\right| = 1$, it follows that $R_x = R_y^{1/3} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

**13.** If we set $y = x^2$, then $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^{2n+1} = \pm\sqrt{y} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} y^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{(-1)^n/3^n}{(-1)^{n+1}/3^{n+1}} \right| = 3$, it follows that $R_x = \sqrt{R_y} = \sqrt{3}$. The open interval of convergence is therefore $-\sqrt{3} < x < \sqrt{3}$.

**14.** Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{(-e)^n/n^2}{(-e)^{n+1}/(n+1)^2} \right| = \frac{1}{e}$, the open interval of convergence is $-1/e < x < 1/e$.

**15.** Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^2/3^{2n}}{(n+1)^2/3^{2n+2}} \right| = 9$, the open interval of convergence is $-9 < x < 9$.

**16.** Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left( \frac{n}{n+1} \right)^n \left( \frac{1}{n+1} \right) = \frac{1}{e}(0) = 0$, the series converges only for $x = 0$.

**17.** Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{1/n^2}{1/(n+1)^2} \right| = 1$, the open interval of convergence is $-11 < x < -9$.

**18.** Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^3 3^n}{(n+1)^3 3^{n+1}} \right| = \frac{1}{3}$, the open interval of convergence is $-1/3 < x < 1/3$.

**19.** If we set $y = x^2$, then $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^2} x^{2n} = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)^2} y^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{3^n/(n+1)^2}{3^{n+1}/(n+2)^2} \right| = 1/3$, it follows that $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is therefore $-1/\sqrt{3} < x < 1/\sqrt{3}$.

**20.** If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty} y^n/5^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{1/5^n}{1/5^{n+1}} \right| = 5$, it follows that $R_x = {R_y}^{1/3} = 5^{1/3}$. The open interval of convergence is therefore $-5^{1/3} < x < 5^{1/3}$.

**21.** Using L'Hôpital's rule, $R = \lim_{n\to\infty} \left| \frac{1/\ln n}{1/\ln(n+1)} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

**22.** Using L'Hôpital's rule, $R = \lim_{n\to\infty} \left| \frac{\frac{1}{n^2 \ln n}}{\frac{1}{(n+1)^2 \ln(n+1)}} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

**23.** Since $R = \lim_{n\to\infty} \left| \frac{(n!)^3/(3n)!}{[(n+1)!]^3/(3n+3)!} \right| = \lim_{n\to\infty} \frac{(n!)^3(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!(n+1)^3(n!)^3} = 27$, the open interval of convergence is $-27 < x < 27$.

**24.** Since $R = \lim_{n\to\infty} \left| \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} \frac{3\cdot 5\cdots(2n+3)}{2\cdot 4\cdots(2n+2)} \right| = \lim_{n\to\infty} \frac{2n+3}{2n+2} = 1$, the open interval of convergence is $-1 < x < 1$.

**25.** Since $R = \lim_{n\to\infty} \left| \frac{\frac{[1\cdot 3\cdots(2n+1)]^2}{2^{2n}(2n)!}}{\frac{[1\cdot 3\cdots(2n+3)]^2}{2^{2n+2}(2n+2)!}} \right| = \lim_{n\to\infty} \frac{4(2n+2)(2n+1)}{(2n+3)^2} = 4$, the open interval of convergence is $-4 < x < 4$.

**26.** $\sum_{n=0}^{\infty} \frac{1}{4^n} x^{3n} = \sum_{n=0}^{\infty} \left( \frac{x^3}{4} \right)^n = \frac{1}{1 - x^3/4} = \frac{4}{4 - x^3}$ provided $\left| \frac{x^3}{4} \right| < 1 \implies |x| < 4^{1/3}$

**27.** $\sum_{n=1}^{\infty} (-e)^n x^n = \sum_{n=1}^{\infty} (-ex)^n = \frac{-ex}{1+ex}$ provided $|-ex| < 1 \implies |x| < 1/e$

**28.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^{2n}}(x-1)^n=\sum_{n=1}^{\infty}\left(\frac{x-1}{9}\right)^n=\frac{\frac{x-1}{9}}{1-\frac{x-1}{9}}=\frac{x-1}{10-x}$ provided $\left|\frac{x-1}{9}\right|<1 \implies |x-1|<9$

**29.** $\displaystyle\sum_{n=2}^{\infty}(x+5)^{2n}=\frac{(x+5)^4}{1-(x+5)^2}$ provided $|(x+5)^2|<1 \implies |x+5|<1$

**30.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n}=\cos(x^2)$ valid for all $x$

**31.** $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n=\sum_{n=0}^{\infty}\frac{1}{n!}(5x)^n=e^{5x}$ valid for all $x$

**32.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}(2n+1)!}x^{2n+2}=x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{x}{3}\right)^{2n+1}=x\sin(x/3)$ valid for all $x$

**33.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{n!}(x+1)^n=\sum_{n=0}^{\infty}\frac{1}{n!}[-3(x+1)]^n=e^{-3(x+1)}$ valid for all $x$

**34.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n=-1+\sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n=-1+e^{-x}$ valid for all $x$

**35.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x+1)^{2n+3}=-(x+1)^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x+1)^{2n+1}=-(x+1)^2\sin(x+1)$ valid for all $x$

**36.** $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}(x-1/2)^n=\sum_{n=0}^{\infty}\frac{1}{n!}(2x-1)^n=e^{2x-1}$ valid for all $x$

**37.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(2n)!}x^{4n+4}=x^4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{x^2}{2}\right)^{2n}=x^4\cos(x^2/2)$ valid for all $x$

**38.** (a) $\displaystyle J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}=1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\frac{x^8}{2^8(4!)^2}-\cdots$

$$J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(n!)(n+1)!}x^{2n+1}=\frac{x}{2}-\frac{x^3}{2^3 2!}+\frac{x^5}{2^5 2!3!}-\frac{x^7}{2^7 3!4!}+\frac{x^9}{2^9 4!5!}-\cdots$$

$$J_m(x)=\frac{x^m}{2^m m!}-\frac{x^{m+2}}{2^{m+2}(m+1)!}+\frac{x^{m+4}}{2^{m+4}2!(m+2)!}-\frac{x^{m+6}}{2^{m+6}3!(m+3)!}+\frac{x^{m+8}}{2^{m+8}4!(m+4)!}-\cdots\cdots$$

(b) $\displaystyle R=\lim_{n\to\infty}\left|\frac{(-1)^n}{2^{2n+m}n!(n+m)!}\frac{2^{2n+m+2}(n+1)!(n+m+1)!}{(-1)^{n+1}}\right|=\lim_{n\to\infty}2^2(n+1)(n+m+1)=\infty$

The interval of convergence is therefore $-\infty<x<\infty$.

**39.** (a) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n$

(b) $\displaystyle R=\lim_{n\to\infty}\left|\frac{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}}{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)}{(n+1)!\,\gamma(\gamma+1)\cdots(\gamma+n)}}\right|=\lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=1$

## EXERCISES 10.5

**1.** $\displaystyle\frac{1}{3x+2}=\frac{1}{2(1+3x/2)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{3x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n3^n}{2^{n+1}}x^n,\quad |-3x/2|<1 \implies |x|<2/3$

**2.** $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right) = \dfrac{1}{4}\sum_{n=0}^{\infty}\left(-\dfrac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{4^{n+1}}x^{2n}, \quad \left|-\dfrac{x^2}{4}\right| < 1 \Longrightarrow |x| < 2$

**3.** Since $\cos x = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}x^{2n}, \quad -\infty < x < \infty,$ it follows that

$$\cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}, \quad -\infty < x < \infty.$$

**4.** Since $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n, \quad -\infty < x < \infty,$ it follows that

$$e^{5x} = \sum_{n=0}^{\infty}\frac{1}{n!}(5x)^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n, \quad -\infty < x < \infty.$$

**5.** Since $f(x) = e^x = e^3\,e^{x-3}$, and the Maclaurin series $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$ converges for all $x$, it follows that

$$e^x = e^3\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n = \sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n, \quad -\infty < x < \infty.$$

**6.** Since $f(x) = e^{1-2x} = e\,e^{-2x}$, and the Maclaurin series $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$ converges for all $x$, it follows that

$$e^{1-2x} = e\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = \sum_{n=0}^{\infty}\frac{e(-1)^n 2^n}{n!}x^n, \quad -\infty < x < \infty.$$

**7.** Since $f(x) = e^{1-2x} = e^{3-2(x+1)} = e^3\,e^{-2(x+1)}$, and the Maclaurin series $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$ converges for all $x$, it follows that

$$e^{1-2x} = e^3\sum_{n=0}^{\infty}\frac{1}{n!}[-2(x+1)]^n = \sum_{n=0}^{\infty}\frac{e^3(-1)^n 2^n}{n!}(x+1)^n, \quad -\infty < x < \infty.$$

**8.**
$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^n + \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[1+(-1)^n]}{n!}x^n \\ &= \frac{1}{2}\left(2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots\right) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}, \quad -\infty < x < \infty\end{aligned}$$

**9.**
$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^n - \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[1-(-1)^n]}{n!}x^n \\ &= \frac{1}{2}\left(2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots\right) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}, \quad -\infty < x < \infty\end{aligned}$$

**10.** This function is its own Maclaurin series.

**11.** Since $f(-2) = 33$, $f'(-2) = -46$, $f''(-2) = 54$, $f'''(-2) = -48$, $f''''(-2) = 24$, and $f^{(n)}(-2) = 0$ for $n \geq 5$, formula 10.17 gives

$$\begin{aligned}f(x) &= 33 - 46(x+2) + \frac{54}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4 \\ &= 33 - 46(x+2) + 27(x+2)^2 - 8(x+2)^3 + (x+2)^4.\end{aligned}$$

**12.** $$\frac{1}{x+3} = \frac{1}{5+(x-2)} = \frac{1}{5\left(1+\dfrac{x-2}{5}\right)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x-2}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-2)^n, \quad \left|-\frac{x-2}{5}\right| < 1 \implies$$
$-3 < x < 7$

**13.** Long division gives

$$\frac{x}{2x+5} = \frac{1}{2} - \frac{5/2}{2x+5} = \frac{1}{2} - \frac{5/2}{2(x-1)+7} = \frac{1}{2} - \frac{5}{14\left[1+\dfrac{2(x-1)}{7}\right]} = \frac{1}{2} - \frac{5}{14}\sum_{n=0}^{\infty}\left[-\frac{2}{7}(x-1)\right]^n$$
$$= \frac{1}{7} + \sum_{n=1}^{\infty}\frac{5(-1)^{n+1}2^{n-1}}{7^{n+1}}(x-1)^n, \quad \left|-\frac{2(x-1)}{7}\right| < 1 \implies -\frac{5}{2} < x < \frac{9}{2}$$

**14.** Long division gives

$$\frac{x^2}{3-4x} = -\frac{x}{4} - \frac{3}{16} + \frac{9/16}{3-4x} = -\frac{1}{4}(x-2) - \frac{11}{16} + \frac{9/16}{-5-4(x-2)} = -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9/80}{1+\dfrac{4(x-2)}{5}}$$
$$= -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9}{80}\sum_{n=0}^{\infty}\left[-\frac{4}{5}(x-2)\right]^n$$
$$= -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9}{80}\left[1 - \frac{4}{5}(x-2) + \sum_{n=2}^{\infty}\frac{(-1)^n 4^n}{5^n}(x-2)^n\right]$$
$$= -\frac{4}{5} - \frac{4}{25}(x-2) + \sum_{n=2}^{\infty}\frac{9(-1)^{n+1}4^{n-2}}{5^{n+1}}(x-2)^n, \quad \left|-\frac{4(x-2)}{5}\right| < 1 \implies \frac{3}{4} < x < \frac{13}{4}$$

**15.** With the binomial expansion 10.33b,

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \cdots, \quad -1 < x \le 1$$
$$= 1 - \frac{x}{2} + \frac{3}{2^2 2!}x^2 - \frac{3\cdot 5}{2^3 3!}x^3 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n n!}x^n$$
$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n)]}{2^n n![2\cdot 4\cdot 6\cdots(2n)]}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}x^n$$

**16.** Term-by-term integration of $\dfrac{1}{1+2x} = \displaystyle\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^n$ gives

$$\frac{1}{2}\ln|1+2x| = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n+1}x^{n+1} + C.$$

Setting $x=0$ gives $C=0$, and therefore $\ln|1+2x| = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+1}}{n+1}x^{n+1}$. Since the radius of convergence of the geometric series is $1/2$, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $-1/2 < x < 1/2$, and the absolute values may be dropped,

$$\ln(1+2x) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+1}}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{n}x^n.$$

**17.** With the binomial expansion 10.33b,

$$\begin{aligned}(1+3x)^{3/2} &= 1+\left(\frac{3}{2}\right)(3x)+\frac{(3/2)(1/2)}{2!}(3x)^2+\frac{(3/2)(1/2)(-1/2)}{3!}(3x)^3+\cdots, \quad -1\le 3x\le 1\\ &= 1+\frac{9}{2}x+\frac{3^3}{2^2 2!}x^2-\frac{3^4}{2^3 3!}x^3+\frac{3^5(1)(3)}{2^4 4!}x^4-\frac{3^6(1)(3)(5)}{2^5 5!}x^5+\cdots\\ &= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)]3^{n+1}}{2^n n!}x^n\\ &= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n-5)(2n-4)]3^{n+1}}{[2\cdot 4\cdots(2n-4)]2^n n!}x^n\\ &= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n(2n-4)!\,3^{n+1}}{2^{2n-2}n!\,(n-2)!}x^n\\ &= 1+\frac{9}{2}x+\sum_{n=2}^{\infty}\frac{(-1)^n(2n-4)!\,3^{n+1}}{2^{2n-2}n!\,(n-2)!}x^n, \quad -\frac{1}{3}\le x\le\frac{1}{3}.\end{aligned}$$

**18.** Termwise integration of

$$\frac{1}{x}=\frac{1}{2+(x-2)}=\frac{1}{2[1+(x-2)/2]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-2}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

gives $\ln|x|=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}+C$. Setting $x=2$ gives $C=\ln 2$, and therefore $\ln|x|=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}$. Since the radius of convergence of the geometric series is 2, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $0<x<4$, and the absolute values may be dropped,

$$\ln x=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x-2)^n.$$

**19.** Termwise integration of

$$\frac{1}{x+3}=\frac{1}{2+(x+1)}=\frac{1}{2[1+(x+1)/2]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x+1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x+1)^n$$

gives $\ln|x+3|=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}+C$. Setting $x=-1$ gives $C=\ln 2$, and therefore $\ln|x+3|=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}$. Since the radius of convergence of the geometric series is 2, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $-3<x<1$, and the absolute values may be dropped,

$$\ln(x+3)=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x+1)^n.$$

**20.** $\frac{1}{x}=\frac{1}{4+(x-4)}=\frac{1}{4[1+(x-4)/4]}=\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x-4}{4}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-4)^n,$ provided $\left|-\frac{x-4}{4}\right|<1 \implies 0<x<8$

**21.** With the binomial expansion 10.33b,

$$\frac{1}{(x+2)^3} = \frac{1}{8(1+x/2)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left[1+(-3)\left(\frac{x}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{1}{8}\left[1-\frac{3x}{2}+\frac{3\cdot 4}{2^3}x^2-\frac{4\cdot 5}{2^4}x^3+\frac{5\cdot 6}{2^5}x^4+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n, \quad \text{valid for } -1<\frac{x}{2}<1 \Longrightarrow -2<x<2.$$

**22.** With the binomial expansion 10.33b,

$$\frac{1}{(2-x)^2} = \frac{1}{[-1-(x-3)]^2} = \frac{1}{[1+(x-3)]^2} = [1+(x-3)]^{-2}$$

$$= 1-2(x-3)+\frac{(-2)(-3)}{2!}(x-3)^2+\frac{(-2)(-3)(-4)}{3!}(x-3)^3+\cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n(n+1)(x-3)^n, \quad \text{provided } -1<x-3<1 \Longrightarrow 2<x<4.$$

**23.** With the binomial expansion 10.33b,

$$\frac{1}{(x+3)^2} = \frac{1}{[4+(x-1)]^2} = \frac{1}{16[1+(x-1)/4]^2} = \frac{1}{16}\left(1+\frac{x-1}{4}\right)^{-2}$$

$$= \frac{1}{16}\left[1-2\left(\frac{x-1}{4}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x-1}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x-1}{4}\right)^3+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{4^{n+2}}(x-1)^n, \quad \text{provided } -1<\frac{x-1}{4}<1 \Longrightarrow -3<x<5.$$

**24.** $$\frac{1}{x^2+8x+15} = \frac{1}{(x+3)(x+5)} = \frac{1/2}{x+3}+\frac{-1/2}{x+5} = \frac{1/6}{1+x/3}-\frac{1/10}{1+x/5}$$

$$= \frac{1}{6}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n-\frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2(3^{n+1})}x^n+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2(5^{n+1})}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(\frac{1}{3^{n+1}}-\frac{1}{5^{n+1}}\right)x^n, \quad \text{valid for } -3<x<3.$$

**25.** Term-by-term integration of $\dfrac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots$ gives

$$\operatorname{Tan}^{-1}x = \left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)+C.$$

Substitution of $x=0$ gives $C=0$, and therefore $\operatorname{Tan}^{-1}x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$. The open interval of convergence is $-1<x<1$.

**26.** With the binomial expansion 10.33b,

$$\sqrt{x+3} = \sqrt{3}\sqrt{1+x/3} = \sqrt{3}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{3}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{x}{3}\right)^2+\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{3}\right)^3+\cdots\right]$$

$$= \sqrt{3}\left[1+\frac{x}{6}-\frac{1}{2^2 3^2 2!}x^2+\frac{(1)(3)}{2^3 3^3 3!}x^3-\frac{(1)(3)(5)}{2^4 3^4 4!}x^4+\cdots\right]$$

$$= \sqrt{3}\left[1+\frac{x}{6}-\frac{x^2}{72}+\sum_{n=3}^{\infty}\frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{2^n 3^n n!}x^n\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}[1\cdot 2\cdot 3\cdot 4\cdots(2n-3)(2n-2)]}{[2\cdot 4\cdot 6\cdots(2n-2)]6^n n!}x^n\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \sum_{n=3}^{\infty} \frac{2(-1)^{n+1}(2n-2)!}{12^n n!\,(n-1)!}x^n\right]$$

$$= \sqrt{3} + \sum_{n=1}^{\infty} \frac{2\sqrt{3}(-1)^{n+1}(2n-2)!}{12^n n!\,(n-1)!}x^n, \quad \text{valid for } -1 \le \frac{x}{3} \le 1 \Longrightarrow |x| \le 3.$$

**27.** With the binomial expansion 10.33b,

$$\sqrt{x+3} = \sqrt{5+(x-2)} = \sqrt{5}\sqrt{1+(x-2)/5}$$

$$= \sqrt{5}\left[1 + \frac{1}{2}\left(\frac{x-2}{5}\right) + \frac{(1/2)(-1/2)}{2!}\left(\frac{x-2}{5}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x-2}{5}\right)^3 + \cdots\right]$$

$$= \sqrt{5}\left[1 + \frac{1}{10}(x-2) - \frac{1}{10^2 2!}(x-2)^2 + \frac{1\cdot 3}{10^3 3!}(x-2)^3 + \cdots\right]$$

$$= \sqrt{5} + \frac{\sqrt{5}}{10}(x-2) + \sum_{n=2}^{\infty} \frac{\sqrt{5}(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{10^n n!}(x-2)^n$$

$$= \sqrt{5} + \frac{\sqrt{5}}{10}(x-2) + \sum_{n=2}^{\infty} \frac{\sqrt{5}(-1)^{n+1}[1\cdot 2\cdot 3\cdots(2n-2)]}{[2\cdot 4\cdot 6\cdots(2n-2)]10^n n!}(x-2)^n$$

$$= \sqrt{5} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{5^{n-1/2}2^{2n-1}n!\,(n-1)!}(x-2)^n, \quad \text{valid for } -1 \le \frac{x-2}{5} \le 1 \Longrightarrow -3 \le x \le 7$$

**28.** With the binomial expansion 10.33b,

$$(1-2x)^{1/3} = [-1-2(x-1)]^{1/3} = -[1+2(x-1)]^{1/3}$$

$$= -\left\{1 + \frac{2(x-1)}{3} + \frac{(1/3)(-2/3)}{2!}[2(x-1)]^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}[2(x-1)])^3 + \cdots\right\}$$

$$= -1 - \frac{2}{3}(x-1) + \frac{2^2 2}{3^2 2!}(x-1)^2 - \frac{2^3(2\cdot 5)}{3^3 3!}(x-1)^3 + \cdots$$

$$= -1 - \frac{2}{3}(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n 2^n[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!}(x-1)^n,$$

$$= -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n[(-1)\cdot 2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!}(x-1)^n,$$

valid for $-1 \le 2(x-1) \le 1 \Longrightarrow 1/2 \le x \le 3/2$.

**29.** With the binomial expansion 10.33b,

$$\frac{x^2}{(1+x^2)^2} = x^2(1+x^2)^{-2} = x^2\left[1 + (-2)(x^2) + \frac{(-2)(-3)}{2!}(x^2)^2 + \frac{(-2)(-3)(-4)}{3!}(x^2)^3 + \cdots\right]$$

$$= x^2 - 2x^4 + 3x^6 - 4x^8 + \cdots$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1}x^{2n}, \quad \text{valid for } -1 < x^2 < 1 \Longrightarrow -1 < x < 1$$

**30.** With the binomial expansion 10.33b,

$$x(1-x)^{1/3} = x\left[1 + \left(\frac{1}{3}\right)(-x) + \frac{(1/3)(-2/3)}{2!}(-x)^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}(-x)^3 + \cdots\right]$$

$$= x - \frac{x^2}{3} - \frac{2}{3^2 2!}x^3 - \frac{(2)(5)}{3^3 3!}x^4 - \frac{(2)(5)(8)}{3^4 4!}x^5 + \cdots$$

$$= x - \frac{x^2}{3} - \sum_{n=3}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-7)}{3^{n-1}(n-1)!} x^n$$

$$= x + \sum_{n=2}^{\infty} \frac{(-1) \cdot 2 \cdot 5 \cdot 8 \cdots (3n-7)}{3^{n-1}(n-1)!} x^n, \quad \text{valid for } -1 \le x \le 1.$$

**31.** We extend the calculations in Example 10.24 to obtain another nonzero term. When we equate coefficients of $x^6$, we obtain $0 = a_6 - a_4/2! + a_2/4! - a_0/6!$, and this implies that $a_6 = 0$. Coefficients of $x^7$ give $-1/7! = a_7 - a_5/2! + a_3/4! - a_1/6! \Longrightarrow a_7 = 17/315$. Consequently,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots,$$

and if we replace $x$ by $2x$, $\quad \tan 2x = 2x + \dfrac{8x^3}{3} + \dfrac{64x^5}{15} + \dfrac{2176x^7}{315} + \cdots.$

**32.** If we set $\sec x = \dfrac{1}{\cos x} = a_0 + a_1 x + a_2 x^2 + \cdots$, then

$$1 = \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right).$$

We now multiply the power series on the right and equate coefficients:

$1$: $\quad 1 = a_0$

$x$: $\quad 0 = a_1$

$x^2$: $\quad 0 = -a_0/2! + a_2 \quad \Longrightarrow \quad a_2 = 1/2$

$x^3$: $\quad 0 = -a_1/2! + a_3 \quad \Longrightarrow \quad a_3 = 0$

$x^4$: $\quad 0 = a_0/4! - a_2/2! + a_4 \quad \Longrightarrow \quad a_4 = 5/24$

$x^5$: $\quad 0 = a_1/4! - a_3/2! + a_5 \quad \Longrightarrow \quad a_5 = 0$

$x^6$: $\quad 0 = -a_0/6! + a_2/4! - a_4/2! + a_6 \quad \Longrightarrow \quad a_6 = 61/720$

Thus, $\sec x = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \dfrac{61}{720}x^6 + \cdots$. Long division could also be used.

**33.** $$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{4!} - \frac{1}{2!3!} + \frac{1}{5!}\right)x^5 + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

**34.** $$\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}\right] = \frac{1}{2}\left[1 + 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}\right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n}, \quad -\infty < x < \infty$$

**35.** $$\frac{1}{x^6 - 3x^3 - 4} = \frac{1}{(x^3-4)(x^3+1)} = \frac{-1/5}{1+x^3} + \frac{1/5}{x^3-4} = \frac{-1/5}{1+x^3} - \frac{1/20}{1 - x^3/4}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty} (-x^3)^n - \frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{x^3}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} -\frac{1}{5}\left[(-1)^n + \frac{1}{4^{n+1}}\right] x^{3n}, \quad \text{valid for } -1 < x < 1.$$

**36.** The Maclaurin series for $\operatorname{Sin}^{-1}(x^2)$ can be obtained by replacing $x$ by $x^2$ in the series for $\operatorname{Sin}^{-1}x$ in Example 10.26:

$$\operatorname{Sin}^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)2^{2n}(n!)^2}(x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)2^{2n}(n!)^2}x^{4n+2}, \quad |x| < 1.$$

**37.** $$\frac{2x^2+4}{x^2+4x+3} = 2 - \frac{8x+2}{(x+3)(x+1)} = 2 - \frac{11}{x+3} + \frac{3}{x+1} = 2 - \frac{11/3}{1+x/3} + \frac{3}{1+x}$$
$$= 2 - \frac{11}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n + 3\sum_{n=0}^{\infty}(-x)^n = \left(2 - \frac{11}{3} + 3\right) + \sum_{n=1}^{\infty}\left[-\frac{11}{3}\left(-\frac{1}{3}\right)^n + 3(-1)^n\right]x^n$$
$$= \frac{4}{3} + \sum_{n=1}^{\infty}(-1)^n\left(3 - \frac{11}{3^{n+1}}\right)x^n, \quad \text{valid for } -1 < x < 1.$$

**38.** If we integrate the series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$ we obtain $-\ln|1-x| = \sum_{n=0}^{\infty}\dfrac{1}{n+1}x^{n+1} + C$. Substitution of $x = 0$ gives $C = 0$, and therefore $\ln|1-x| = \sum_{n=0}^{\infty}\dfrac{-1}{n+1}x^{n+1} = \sum_{n=1}^{\infty}-\dfrac{1}{n}x^n$. The open interval of convergence is $-1 < x < 1$ so that absolute values may be dropped. If we replace $x$ by $x/\sqrt{2}$ and $-x/\sqrt{2}$, we find

$$f(x) = \ln(1 + x/\sqrt{2}) - \ln(1 - x/\sqrt{2}) = \sum_{n=1}^{\infty}-\frac{1}{n}\left(-\frac{x}{\sqrt{2}}\right)^n - \sum_{n=1}^{\infty}-\frac{1}{n}\left(\frac{x}{\sqrt{2}}\right)^n$$
$$= \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n2^{n/2}} + \frac{1}{n2^{n/2}}\right]x^n = \sum_{n=1}^{\infty}\left[\frac{1+(-1)^{n+1}}{n2^{n/2}}\right]x^n.$$

When $n$ is even the coefficient of $x^n$ is zero, and therefore

$$f(x) = \sum_{n=0}^{\infty}\frac{2}{(2n+1)2^{(2n+1)/2}}x^{2n+1} = \sum_{n=0}^{\infty}\frac{\sqrt{2}}{(2n+1)2^n}x^{2n+1}.$$

Since the added series both have open interval of convergence $-\sqrt{2} < x < \sqrt{2}$, this is the open interval of convergence for the combined series.

**39.** If $\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} b_n(x-c)^n$, then $\sum_{n=0}^{\infty}(a_n - b_n)(x-c)^n = 0$. The right side of this equation is the Maclaurin series for the function identically equal to zero, and as such, its coefficients must all be zero; that is, $a_n - b_n = 0$ for all $n$.

**40.** The right side of this equation is the Maclaurin series for the function identically equal to zero, and as such, its coefficients must all be zero; that is, $a_n = 0$ for all $n$.

**41.** $\sum_{n=0}^{\infty}P_n(t) = \sum_{n=0}^{\infty}\dfrac{1}{n!}\left(\dfrac{t}{30}\right)^n e^{-t/30} = e^{-t/30}\sum_{n=0}^{\infty}\dfrac{1}{n!}\left(\dfrac{t}{30}\right)^n = e^{-t/30}(e^{t/30}) = 1$ The sum represents the probability that either nobody, or just one person, or two people, or three people, etc., drink from the fountain. Since one of these situations must occur, the probability is one.

**42.** (a) $\sum_{n=1}^{\infty}np(1-p)^{n-1} = p\sum_{n=1}^{\infty}n(1-p)^{n-1}$ If we differentiate the series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty}x^n$, $|x| < 1$, term-by-term, we obtain $\dfrac{1}{(1-x)^2} = \sum_{n=0}^{\infty}nx^{n-1} = \sum_{n=1}^{\infty}nx^{n-1}, \quad |x| < 1$. We now substitute $x = 1 - p$ into this result, $\dfrac{1}{[1-(1-p)]^2} = \sum_{n=1}^{\infty}n(1-p)^{n-1}$. Multiplication by $p$ gives $\dfrac{1}{p} = \sum_{n=1}^{\infty}np(1-p)^{n-1}$.

(b) The probability of throwing a six is $p = 1/6$, and therefore $\sum_{n=1}^{\infty}np(1-p)^{n-1} = \dfrac{1}{1/6} = 6$.

**43.** $$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\int_0^x \left[\sum_{n=0}^{\infty}\frac{1}{n!}(-t^2)^n\right]dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n}\,dt$$
$$= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{t^{2n+1}}{2n+1}\right\}_0^x = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}$$

**44.** Integrating the Maclaurin series for $\cos(\pi t^2/2)$ (see Example 10.21) term-by-term gives
$$C(x) = \int_0^x \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi t^2}{2}\right)^{2n}\right]dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\pi^{2n}}{2^{2n}}\int_0^x t^{4n}\,dt$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\pi^{2n}}{2^{2n}}\left\{\frac{t^{4n+1}}{4n+1}\right\}_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(4n+1)2^{2n}(2n)!}x^{4n+1},$$
valid for $-\infty < x < \infty$. A similar procedure leads to the Maclaurin series for $S(x)$.

**45.** With the binomial expansion 10.33b,
$$\frac{x}{(4+3x)^2} = \frac{x}{16}\left(1+\frac{3x}{4}\right)^{-2} = \frac{x}{16}\left[1 - 2\left(\frac{3x}{4}\right) + \frac{(-2)(-3)}{2!}\left(\frac{3x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{3x}{4}\right)^3 + \cdots\right]$$
$$= \frac{x}{16}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n(n+1)}{4^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n(n+1)}{4^{n+2}}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^{n-1}n}{4^{n+1}}x^n.$$
But the coefficient of $x^n$ in the Maclaurin series is $f^{(n)}(0)/n!$, and therefore
$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}3^{n-1}n}{4^{n+1}} \quad\Longrightarrow\quad f^{(n)}(0) = \frac{(-1)^{n+1}3^{n-1}n\,n!}{4^{n+1}}.$$

**46.** The Maclaurin series for $f(x) = xe^{-2x}$ is
$$xe^{-2x} = x\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{n-1}}{(n-1)!}x^n.$$
But the coefficient of $x^n$ in the Maclaurin series is $f^{(n)}(0)/n!$, and therefore
$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}2^{n-1}}{(n-1)!} \quad\Longrightarrow\quad f^{(n)}(0) = \frac{(-1)^{n+1}2^{n-1}n!}{(n-1)!} = n(-1)^{n+1}2^{n-1}.$$

**47.** The Taylor series for $f(x) = 1/(3+x)$ about $x = 2$ is
$$\frac{1}{3+x} = \frac{1}{5+(x-2)} = \frac{1}{5[1+(x-2)/5]} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x-2}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-2)^n.$$
But the coefficient of $(x-2)^n$ in the Taylor series is $f^{(n)}(2)/n!$, and therefore
$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{5^{n+1}} \quad\Longrightarrow\quad f^{(n)}(2) = \frac{(-1)^n n!}{5^{n+1}}.$$

**48.** The Taylor series for $f(x) = xe^{-x}$ about $x = 2$ is
$$xe^{-x} = [(x-2)+2]e^{-(x-2)-2} = e^{-2}[2+(x-2)]\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-2)^n$$
$$= e^{-2}\left[\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!}(x-2)^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-2)^{n+1}\right]$$
$$= e^{-2}\left[\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!}(x-2)^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!}(x-2)^n\right]$$
$$= e^{-2}\left[2 + \sum_{n=1}^{\infty}\frac{(-1)^n(2-n)}{n!}(x-2)^n\right].$$

But the coefficient of $(x-2)^n$ in the Taylor series is $f^{(n)}(2)/n!$, and therefore

$$\frac{f^{(n)}(2)}{n!}=\frac{(-1)^n(2-n)e^{-2}}{n!} \quad\Longrightarrow\quad f^{(n)}(2)=\frac{(-1)^n(2-n)n!}{e^2 n!}=\frac{(n-2)(-1)^{n+1}}{e^2}.$$

**49.** Since the Maclaurin series for $x^2\sin 2x$

$$x^2\sin 2x=x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+3}$$

contains only odd powers of $x$, the even derivatives of $x^2\sin 2x$ must all be zero.

**50.** Since the Maclaurin series for $e^{-x^2}$, namely, $e^{-x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}(-x^2)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}$ contains only even powers of $x$, the odd derivatives of $e^{-x^2}$ must all be zero.

**51.** Using the definition of $J_m(x)$ as the Maclaurin series in Exercise 38 of Section 10.4, we may write

$$\begin{aligned}
2m\,J_m(x)-x\,J_{m-1}(x)&=2m\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m}n!(n+m)!}x^{2n+m}-x\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m-1}\\
&=\sum_{n=0}^{\infty}\frac{m(-1)^n}{2^{2n+m-1}n!(n+m)!}x^{2n+m}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n(m-n-m)}{2^{2n+m-1}n!\,(n+m)!}x^{2n+m}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{2n+m-1}(n-1)!\,(n+m)!}x^{2n+m}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m+1}n!\,(n+m+1)!}x^{2n+m+2}=x\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m+1}n!\,(n+m+1)!}x^{2n+m+1}\\
&=x\,J_{m+1}(x).
\end{aligned}$$

**52.** Using the definition of $J_m(x)$ as the Maclaurin series in Exercise 38 of Section 10.4, we may write

$$J_{m-1}(x)-J_{m+1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m-1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m+1}n!(n+m+1)!}x^{2n+m+1}.$$

We lower $n$ by 1 in the second summation, and separate out the first term in the first summation,

$$\begin{aligned}
J_{m-1}(x)-J_{m+1}(x)&=\frac{1}{2^{m-1}(m-1)!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m-1}\\
&\quad+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m)!}x^{2n+m-1}\\
&=\frac{1}{2^{m-1}(m-1)!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m-1)!}\left(\frac{1}{n}+\frac{1}{n+m}\right)x^{2n+m-1}\\
&=\frac{1}{2^{m-1}(m-1)!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m-1)!}\left[\frac{2n+m}{n(n+m)}\right]x^{2n+m-1}\\
&=\frac{m}{2^{m-1}m!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(2n+m)(-1)^n}{2^{2n+m-1}n!(n+m)!}x^{2n+m-1}\\
&=\sum_{n=0}^{\infty}\frac{(2n+m)(-1)^n}{2^{2n+m-1}n!(n+m)!}x^{2n+m-1}.
\end{aligned}$$

Term-by-term differentiation of the series for $J_m(x)$ gives $J_m'(x)=\sum_{n=0}^{\infty}\frac{(2n+m)(-1)^n}{2^{2n+m}n!(n+m)!}x^{2n+m-1}$.

Hence, $J_{m-1}(x)-J_{m+1}(x)=2J_m'(x)$.

**53.** With the binomial expansion 10.33b,

$$\begin{aligned}\frac{1}{\sqrt{1-2\mu x+x^2}} &= 1-\frac{1}{2}(x^2-2\mu x)+\frac{(-1/2)(-3/2)}{2!}(x^2-2\mu x)^2+\frac{(-1/2)(-3/2)(-5/2)}{3!}(x^2-2\mu x)^3+\cdots\\ &= 1+\frac{1}{2}(2\mu x-x^2)+\frac{3}{8}(4\mu^2x^2-4\mu x^3+x^4)+\frac{5}{16}(8\mu^3x^3-12\mu^2x^4+6\mu x^5-x^6)+\cdots\\ &= 1+(\mu)x+\left(-\frac{1}{2}+\frac{3\mu^2}{2}\right)x^2+\left(-\frac{3\mu}{2}+\frac{5\mu^3}{2}\right)x^3+\cdots.\end{aligned}$$

Thus, $P_0(\mu)=1$, $P_1(\mu)=\mu$, $P_2(\mu)=(3\mu^2-1)/2$, and $P_3(\mu)=(5\mu^3-3\mu)/2$.

**54.** (a) If we substitute the Maclaurin series for $e^x$ into $x=(e^x-1)\left(1+B_1x+\dfrac{B_2}{2!}x^2+\cdots\right)$,

$$\begin{aligned}x &= \left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1\right]\left(1+B_1x+\frac{B_2x^2}{2!}+\cdots\right)\\ &= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots\right)\left(1+B_1x+\frac{B_2x^2}{2!}+\frac{B_3x^3}{3!}+\frac{B_4x^4}{4!}+\frac{B_5x^5}{5!}+\cdots\right).\end{aligned}$$

When we multiply the series on the right and equate coefficients of powers of $x$ left and right:

$x$: $1=1$

$x^2$: $0=\dfrac{1}{2!}+B_1 \implies B_1=-\dfrac{1}{2}$

$x^3$: $0=\dfrac{1}{3!}+\dfrac{B_1}{2!}+\dfrac{B_2}{2!} \implies B_2=\dfrac{1}{6}$

$x^4$: $0=\dfrac{1}{4!}+\dfrac{B_1}{3!}+\dfrac{B_2}{(2!)^2}+\dfrac{B_3}{3!} \implies B_3=0$

$x^5$: $0=\dfrac{1}{5!}+\dfrac{B_1}{4!}+\dfrac{B_2}{2!3!}+\dfrac{B_3}{2!3!}+\dfrac{B_4}{4!} \implies B_4=-\dfrac{1}{30}$

$x^6$: $0=\dfrac{1}{6!}+\dfrac{B_1}{5!}+\dfrac{B_2}{2!4!}+\dfrac{B_3}{(3!)^2}+\dfrac{B_4}{2!4!}+\dfrac{B_5}{5!} \implies B_5=0$

(b) Suppose we set
$$\begin{aligned}f(x) &= \frac{x}{e^x-1}-1-B_1x=\frac{x}{e^x-1}-1+\frac{x}{2}=\frac{2x-2(e^x-1)+x(e^x-1)}{2(e^x-1)}\\ &= \frac{xe^x-2e^x+x+2}{2(e^x-1)}=\frac{B_2}{2!}x^2+\frac{B_3}{3!}x^3+\cdots.\end{aligned}$$

Since

$$\begin{aligned}f(-x) &= \frac{-x}{e^{-x}-1}-1-\frac{x}{2}=\frac{xe^x}{e^x-1}-1-\frac{x}{2}\\ &= \frac{2xe^x-2(e^x-1)-x(e^x-1)}{2(e^x-1)}=\frac{xe^x-2e^x+x+2}{2(e^x-1)}=f(x),\end{aligned}$$

$f(x)$ is an even function. But the Maclaurin series for $f(x)$ can represent an even function only if all odd powers are absent. In other words, $0=B_3=B_5=\cdots$.

**55.** $e^{x(t-1/t)/2}=e^{xt/2}\,e^{-x/(2t)}=\left[\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{xt}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x}{2t}\right)^n\right]=\left[\sum_{n=0}^{\infty}\frac{(x/2)^n}{n!}t^n\right]\left[\sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!}\left(\frac{1}{t}\right)^n\right].$

When these series are multiplied together, the coefficient of $t^n$ is

$$\begin{aligned}\frac{(x/2)^n}{n!}+\frac{(x/2)^{n+1}}{(n+1)!}\frac{(-x/2)}{1!}+\frac{(x/2)^{n+2}}{(n+2)!}\frac{(-x/2)^2}{2!}+\cdots &= \sum_{m=0}^{\infty}\frac{(x/2)^{n+m}}{(n+m)!}\frac{(-x/2)^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+n}m!\,(n+m)!}x^{2m+n}=J_n(x).\end{aligned}$$

**EXERCISES 10.6**

**1.** The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n}{n+1}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty} nx^{n-1}$, then term-by-term integration gives

$$\int S(x)\,dx + C = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x},$$

since the series is geometric. Differentiation now gives $S(x) = \dfrac{(1-x)(1) - x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$.

**2.** The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n(n-1)}{(n+1)n}\right| = 1$. If we set $S(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$, then term-by-term integration gives $\int S(x)\,dx + C = \sum_{n=2}^{\infty} nx^{n-1}$. A second integration leads to

$$\int\left[\int S(x)\,dx + C\right]dx + D = \sum_{n=2}^{\infty} x^n = \frac{x^2}{1-x},$$

since the series is geometric. Differentiation now gives

$$\int S(x)\,dx + C = \frac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}.$$

A second differentiation provides $S(x)$,

$$S(x) = \frac{(1-x)^2(2-2x) - (2x-x^2)2(1-x)(-1)}{(1-x)^4} = \frac{2}{(1-x)^3}.$$

**3.** The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n+1}{n+2}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty}(n+1)x^{n-1}$, then $x\,S(x) = \sum_{n=1}^{\infty}(n+1)x^n$. Term-by-term integration gives

$$\int x\,S(x)\,dx + C = \sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x},$$

since the series is geometric. Differentiation now gives

$$x\,S(x) = \frac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2} \qquad\Longrightarrow\qquad S(x) = \frac{2-x}{(1-x)^2}.$$

**4.** The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n^2}{(n+1)^2}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty} n^2x^{n-1}$, then term-by-term integration gives $\int S(x)\,dx + C = \sum_{n=1}^{\infty} nx^n$. When $x \neq 0$, we can divide by $x$, $\dfrac{1}{x}\int S(x)\,dx + \dfrac{C}{x} = \sum_{n=1}^{\infty} nx^{n-1}$. Integration now gives,

$$\int\left[\frac{1}{x}\int S(x)\,dx\right]dx + C\ln|x| + D = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

If we now differentiate, $\displaystyle \frac{1}{x}\int S(x)\,dx + \frac{C}{x} = \frac{(1-x)(1) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$. Multiplication by $x$ and a further differentiation gives

$$S(x) = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{(1-x)^2(1) - x(2)(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3}.$$

Since the sum of the series at $x = 0$ is 1, and this is $S(0)$, the formula $S(x) = (x+1)/(1-x)^3$ can be used for all $x$ in $|x| < 1$.

**5.** If we divide the series into two parts, $\displaystyle\sum_{n=1}^{\infty}(n^2+2n)x^n = \sum_{n=1}^{\infty} n^2x^n + 2\sum_{n=1}^{\infty} nx^n$, the first series is $x$ times that in Exercise 4, and the second is $x$ times that in Exercise 1. Hence,

$$\sum_{n=1}^{\infty}(n^2+2n)x^n = \frac{x(x+1)}{(1-x)^3} + \frac{2x}{(1-x)^2} = \frac{3x - x^2}{(1-x)^3}.$$

**6.** The radius of convergence of the series is $\displaystyle R = \lim_{n\to\infty}\left|\frac{1/(n+1)}{1/(n+2)}\right| = 1$. If we set $\displaystyle S(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}x^n$, then $\displaystyle x\,S(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$. Term-by-term differentiation gives $\displaystyle \frac{d}{dx}[x\,S(x)] = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, since the series is geometric. We now integrate,

$$x\,S(x) = \int \frac{1}{1-x}dx = -\ln(1-x) + C.$$

Substitution of $x = 0$ gives $C = 0$, and therefore $S(x) = -\dfrac{1}{x}\ln(1-x)$. This is valid for $-1 < x < 1$, but not at $x = 0$. It is interesting to note, however, that the limit of $S(x)$ as $x$ approaches zero is 1 and this is the sum of the series at $x = 0$.

**7.** If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = \pm\sqrt{y}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}y^n$. The radius of convergence of this series is $\displaystyle R_y = \lim_{n\to\infty}\left|\frac{(-1)^n/(2n+1)}{(-1)^{n+1}/(2n+3)}\right| = 1$. The radius of convergence of the original series is therefore $R_x = 1$. If we set $\displaystyle S(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$, then term-by-term differentiation gives $\displaystyle S'(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$, since the series is geometric. Integration now gives $S(x) = \mathrm{Tan}^{-1}x + C$. Since $S(0) = 0$, it follows that $C = 0$, and $S(x) = \mathrm{Tan}^{-1}x$.

**8.** If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}y^n$. The radius of convergence of this series is $\displaystyle R_y = \lim_{n\to\infty}\left|\frac{(-1)^n/n}{(-1)^{n+1}/(n+1)}\right| = 1$. The radius of convergence of the original series is therefore $R_x = 1$. If we set $\displaystyle S(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n}$, then term-by-term differentiation gives

$$S'(x) = \sum_{n=1}^{\infty}2(-1)^n x^{2n-1} = \frac{-2x}{1+x^2},$$

since the series is geometric. Integration now leads to $S(x) = -\ln(1+x^2) + C$. Since $S(0) = 0$, it follows that $C = 0$, and $S(x) = -\ln(1+x^2)$.

**9.** If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=2}^{\infty} n3^n x^{2n} = \sum_{n=2}^{\infty} n3^n y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{n3^n}{(n+1)3^{n+1}}\right| = 1/3$. The radius of convergence of the original series is therefore $R_x = 1/\sqrt{3}$. If we set $S(x) = \displaystyle\sum_{n=2}^{\infty} n3^n x^{2n}$, then $\dfrac{S(x)}{x} = \displaystyle\sum_{n=2}^{\infty} n3^n x^{2n-1}$, provided $x \neq 0$. Term-by-term integration of this equation gives

$$\int \frac{S(x)}{x}dx = \sum_{n=2}^{\infty}\frac{3^n}{2}x^{2n} = \frac{9x^4/2}{1-3x^2},$$

since the series is geometric. Differentiation now gives

$$\frac{S(x)}{x} = \frac{9}{2}\left[\frac{(1-3x^2)(4x^3) - x^4(-6x)}{(1-3x^2)^2}\right] = \frac{9(4x^3-6x^5)}{2(1-3x^2)^2} \implies S(x) = \frac{9x^4(2-3x^2)}{(1-3x^2)^2}.$$

Since the sum of the series at $x = 0$ is 0, and this is $S(0)$, the formula for $S(x)$ can be used for all $x$ in $|x| < 1/\sqrt{3}$.

**10.** The radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{(n+1)/(n+2)}{(n+2)/(n+3)}\right| = 1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)x^n$, and integrate, $\displaystyle\int S(x)\,dx = \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+1} + C$. Multiplication by $x$ gives $x\displaystyle\int S(x)\,dx = \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+2} + Cx$. Differentiation now gives

$$\frac{d}{dx}\left[x\int S(x)\,dx\right] = \sum_{n=0}^{\infty}x^{n+1} + C = \frac{x}{1-x} + C,$$

since the series is geometric. Integration now yields

$$x\int S(x)\,dx = \int\frac{x}{1-x}dx + Cx + D = -x - \ln|1-x| + Cx + D.$$

If we set $x = 0$ in this equation we find that $D = 0$. When we drop absolute values and divide by $x$,

$$\int S(x)\,dx = -1 - \frac{1}{x}\ln(1-x) + C, \quad x \neq 0.$$

When we differentiate this equation, we obtain $S(x) = \dfrac{1}{x^2}\ln(1-x) + \dfrac{1}{x(1-x)}$. This formula can only be used for values of $x$ in the interval $-1 < x < 1$, but not $x = 0$. The sum at $x = 0$ is $1/2$.

**11.** The radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{(n+1)/n!}{(n+2)/(n+1)!}\right| = \infty$. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n!}\right)x^n$, and integrate,

$$\int S(x)\,dx = \sum_{n=1}^{\infty}\frac{x^{n+1}}{n!} + C = x\sum_{n=1}^{\infty}\frac{x^n}{n!} + C = x(e^x - 1) + C.$$

Differentiation now gives

$$S(x) = (e^x - 1) + x(e^x) = (x+1)e^x - 1.$$

**12.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)!}x^{2n+1} = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = x\cos x$

**13.** If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!}x^{2n} = \pm\sqrt{y}\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n(n+2)/(2n)!}{(-1)^{n+1}(n+3)/(2n+2)!}\right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!}x^{2n}$, and multiply by $x^3$, $x^3S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!}x^{2n+3}$. Integration now gives

$$\int x^3S(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!(2n+4)}x^{2n+4} + C = \frac{x^4}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + C = \frac{x^4}{2}\cos x + C.$$

We now differentiate to get

$$x^3S(x) = 2x^3\cos x - \frac{x^4}{2}\sin x \qquad \Longrightarrow \qquad S(x) = 2\cos x - \frac{x}{2}\sin x.$$

**14.** If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!}x^{2n} = \sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(2n+3)2^n/n!}{(2n+5)2^{n+1}/(n+1)!}\right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!}x^{2n}$, and multiply by $x^2$, $x^2S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!}x^{2n+2}$. Integration now gives

$$\int x^2S(x)\,dx = \sum_{n=1}^{\infty} \frac{2^n}{n!}x^{2n+3} + C = x^3\sum_{n=1}^{\infty} \frac{1}{n!}(2x^2)^n + C = x^3(e^{2x^2} - 1) + C.$$

We now differentiate to get

$$x^2S(x) = 3x^2(e^{2x^2} - 1) + x^3(4xe^{2x^2}) \qquad \Longrightarrow \qquad S(x) = (4x^2 + 3)e^{2x^2} - 3.$$

**15.** If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!}x^{2n+1} = \pm\sqrt{y}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n-1)/(2n)!}{(-1)^{n+2}(2n+1)/(2n+2)!}\right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!}x^{2n+1}$, and divide by $x^3$, $\dfrac{S(x)}{x^3} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!}x^{2n-2}$, $\quad x \neq 0$. Integration now gives

$$\int \frac{S(x)}{x^3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}x^{2n-1} + C = -\frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + C = -\frac{1}{x}\cos x + C.$$

We now differentiate to get

$$\frac{S(x)}{x^3} = \frac{1}{x^2}\cos x + \frac{1}{x}\sin x \qquad \Longrightarrow \qquad S(x) = x\cos x + x^2\sin x.$$

This gives the sum of the series at $x = 0$ also.

**EXERCISES 10.7**

**1.** Taylor's remainder formula for $e^x$ and $c = 0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $0 < z_3 < x$. Since $x \le 0.01$, we can say that

$$R_3 < e^x\frac{x^4}{24} \le e^{0.01}\frac{(0.01)^4}{24} = 4.2 \times 10^{-10}.$$

**2.** Taylor's remainder formula for $e^x$ and $c = 0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $0 < z_3 < x$. Since $x < 0.01$, we can say that

$$R_3 < e^x\frac{x^4}{24} < e^{0.01}\frac{(0.01)^4}{24} = 4.2 \times 10^{-10}.$$

**3.** Taylor's remainder formula for $e^x$ and $c = 0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $x < z_3 < 0$. Since $-0.01 \le x < 0$, we can say that

$$|R_3| < e^0\frac{|x|^4}{24} \le \frac{|-0.01|^4}{24} = 4.2 \times 10^{-10}.$$

**4.** According to Exercise 2, a maximum error on $0 \le x \le 0.01$ is $4.2 \times 10^{-10}$. For $-0.01 \le x < 0$, $R_3 = e^{z_3}\dfrac{x^4}{24}$ where $x < z_3 < 0$. Since $x \ge -0.01$, it follows that

$$|R_3| < e^0\frac{|x|^4}{24} \le \frac{|-0.01|^4}{24} < 4.2 \times 10^{-10}.$$

**5.** Taylor's remainder formula for $\sin x$ and $c = 0$ gives $\sin x = x - \dfrac{x^3}{3!} + R_4$, where $R_4 = \dfrac{d^5}{dx^5}\sin x_{|x=z_4}\dfrac{x^5}{5!} = (\cos z_4)\dfrac{x^5}{120}$, and $0 < z_4 < x$. Since $0 \le x \le 1$, we can say that

$$R_4 < (1)\frac{x^5}{120} \le \frac{(1)^5}{120} = \frac{1}{120}.$$

**6.** Taylor's remainder formula for $\cos x$ and $c = 0$ gives $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + R_5$, where $R_5 = \dfrac{d^6}{dx^6}\cos x_{|x=z_5}\dfrac{x^6}{6!} = -(\cos z_5)\dfrac{x^6}{6!}$, and $z_5$ is between 0 and $x$. Since $|x| \le 0.1$, we can say that

$$|R_5| < (1)\frac{|x|^6}{6!} \le \frac{(0.1)^6}{6!} < 1.4 \times 10^{-9}.$$

**7.** The first four derivatives of $f(x) = \ln(1-x)$ are $f'(x) = -1/(1-x)$, $f''(x) = -1/(1-x)^2$, $f'''(x) = -2/(1-x)^3$, and $f''''(x) = -6/(1-x)^4$. Taylor's remainder formula for $\ln(1-x)$ and $c = 0$ gives $\ln(1-x) = -x - x^2/2 - x^3/3 + R_3(x)$, where $R_3(x) = f''''(z_3)\dfrac{x^4}{4!} = \dfrac{-x^4}{4(1-z_3)^4}$, and $0 < z_3 < x$. Since $0 \le x \le 0.01$, we can say that

$$|R_3| < \frac{x^4}{4(1-x)^4} \le \frac{(0.01)^4}{4(1-0.01)^4} < 2.7 \times 10^{-9}.$$

**8.** The first four derivatives of $f(x) = 1/(1-x)^3$ are $f'(x) = 3/(1-x)^4$, $f''(x) = 12/(1-x)^5$, $f'''(x) = 60/(1-x)^6$, and $f''''(x) = 360/(1-x)^7$. Taylor's remainder formula for $1/(1-x)^3$ and $c = 0$ gives $\dfrac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + R_3(x)$, where $R_3(x) = f''''(z_3)\dfrac{x^4}{4!} = \dfrac{15x^4}{(1-z_3)^7}$, and $z_3$ is between 0 and $x$. Since $|x| < 0.2$, we can say that

$$|R_3| < \frac{15|x|^4}{(1-0.2)^7} < \frac{15(0.2)^4}{(1-0.2)^7} < 0.115.$$

**9.** Taylor's remainder formula for $\sin 3x$ and $c = 0$ gives $\sin 3x = 3x - \dfrac{9x^3}{2} + \dfrac{81x^5}{40} + R_6$, where $R_6 = \dfrac{d^7}{dx^7}\sin 3x_{|x=z_6}\dfrac{x^7}{7!} = -3^7(\cos 3z_6)\dfrac{x^7}{7!}$, and $z_6$ is between 0 and $x$. Since $|x| < \pi/100$, we can say that

$$|R_6| < 3^7(1)\frac{|x|^7}{7!} < 3^7\frac{(\pi/100)^7}{7!} < 1.4 \times 10^{-11}.$$

**10.** The first five derivatives of $f(x) = \ln x$ are $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$, $f''''(x) = -6/x^4$, and $f'''''(x) = 24/x^5$. Taylor's remainder formula with $c = 1$ gives

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + R_4,$$

where $R_4 = f^{(5)}(z_4)\dfrac{(x-1)^5}{5!} = \dfrac{24}{z_4^5}\dfrac{(x-1)^5}{5!} = \dfrac{(x-1)^5}{5z_4^5}$ and $z_4$ is between 1 and $x$. Since $1/2 \le x \le 3/2$, we can say that

$$|R_4| < \frac{|x-1|^5}{5(1/2)^5} \le \frac{(1/2)^5}{5(1/2)^5} = 0.2.$$

**11.** Taylor's remainder formula for $\sin x$ gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{d^n}{dx^n}(\sin x)_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\dfrac{x^{n+1}}{(n+1)!}$ and $z_n$ is between 0 and $x$. Therefore

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{1}{x}R_n(0,x).$$

When we take definite integrals,

$$\begin{aligned}\int_0^1 \frac{\sin x}{x}dx &= \int_0^1 \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{1}{x}R_n(0,x)\right] dx\\ &= \left\{x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots\right\}_0^1 + \int_0^1 \frac{1}{x}R_n(0,x)\,dx,\\ &= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \cdots + \int_0^1 \frac{1}{x}R_n(0,x)\,dx.\end{aligned}$$

Now, $\left|\displaystyle\int_0^1 \frac{1}{x}R_n(0,x)\,dx\right| \le \displaystyle\int_0^1 \left|\frac{1}{x}\frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}\right| dx$. Since $\left|\dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\right| \le 1$, it follows that

$$\left|\int_0^1 \frac{1}{x}R_n(0,x)\,dx\right| \le \int_0^1 \frac{x^n}{(n+1)!}dx = \left\{\frac{x^{n+1}}{(n+1)(n+1)!}\right\}_0^1 = \frac{1}{(n+1)(n+1)!}.$$

When $n = 6$, this is less than 0.000 029. Hence, if we approximate the integral with the first three terms, namely, $1 - \dfrac{1}{3\cdot 3!} + \dfrac{1}{5\cdot 5!} = 0.946\,111$, then we can say that

$$0.946\,111 - 0.000\,029 < \int_0^1 \frac{\sin x}{x} dx < 0.946\,111 + 0.000\,029;$$

that is, $0.946\,082 < \displaystyle\int_0^1 \frac{\sin x}{x} dx < 0.946\,140$. To three decimals, then, the value of the integral is 0.946.

**12.** If we set $u = x^2$ and $du = 2x\,dx$, then $\displaystyle\int_0^{1/2} \cos(x^2)\,dx = \frac{1}{2}\int_0^{1/4} \frac{\cos u}{\sqrt{u}} du$. Taylor's remainder formula for $\cos u$ gives

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + \frac{d^n(\cos u)}{du^n}_{|u=0} \frac{u^n}{n!} + R_n(0,u),$$

where $R_n(0,u) = \dfrac{d^{n+1}(\cos u)}{du^{n+1}}_{|u=z_n} \dfrac{u^{n+1}}{(n+1)!}$. Consequently,

$$\begin{aligned}
\int_0^{1/2} \cos(x^2)\,dx &= \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} \left[1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + R_n(0,u)\right] du \\
&= \frac{1}{2}\int_0^{1/4} \left[\frac{1}{\sqrt{u}} - \frac{u^{3/2}}{2!} + \frac{u^{7/2}}{4!} - \cdots + \frac{1}{\sqrt{u}} R_n(0,u)\right] du \\
&= \frac{1}{2}\left\{2\sqrt{u} - \frac{2u^{5/2}}{5 \cdot 2!} + \frac{2u^{9/2}}{9 \cdot 4!} - \cdots\right\}_0^{1/4} + \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0,u)\,du \\
&= \frac{1}{2} - \frac{1}{5 \cdot 2^5 \cdot 2!} + \frac{1}{9 \cdot 2^9 \cdot 4!} - \cdots + \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0,u)\,du.
\end{aligned}$$

Now,

$$\begin{aligned}
\left|\frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0,u)\,du\right| &\le \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} |R_n(0,u)|\,du \le \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} \frac{u^{n+1}}{(n+1)!} du \\
&= \frac{1}{2}\int_0^{1/4} \frac{u^{n+1/2}}{(n+1)!} du = \frac{1}{2(n+1)!}\left\{\frac{u^{n+3/2}}{n+3/2}\right\}_0^{1/4} = \frac{1}{(2n+3)(n+1)!4^{n+3/2}}.
\end{aligned}$$

When $n = 2$, this is less than $1.9 \times 10^{-4}$. Hence, if we approximate the integral with the first two terms, namely, $\dfrac{1}{2} - \dfrac{1}{5 \cdot 2^5 \cdot 2!} = \dfrac{159}{320}$, then we can say that

$$\frac{159}{320} - 0.000\,19 < \int_0^{1/2} \cos(x^2)\,dx < \frac{159}{320} + 0.000\,19,$$

that is, $0.496\,685 < \displaystyle\int_0^{1/2} \cos(x^2)\,dx < 0.497\,065$. To three decimals, the value of the integral is 0.497.

**13.** Taylor's remainder formula for $\sin x$ gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{d^n}{dx^n}(\sin x)_{|x=0} \frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!}$ and $z_n$ is between 0 and $x$. Therefore

$$x^{11}\sin x = x^{12} - \frac{x^{14}}{3!} + \frac{x^{16}}{5!} - \cdots + x^{11}R_n(0,x).$$

When we take definite integrals,

$$\begin{aligned}\int_{-1}^{1} x^{11}\sin x\,dx &= \int_{-1}^{1}\left[x^{12}-\frac{x^{14}}{3!}+\frac{x^{16}}{5!}-\cdots+x^{11}R_n(0,x)\right]dx\\ &= \left\{\frac{x^{13}}{13}-\frac{x^{15}}{15\cdot 3!}+\frac{x^{17}}{17\cdot 5!}-\cdots\right\}_{-1}^{1}+\int_{-1}^{1}x^{11}R_n(0,x)\,dx,\\ &= \frac{2}{13}-\frac{2}{15\cdot 3!}+\frac{2}{17\cdot 5!}-\cdots+\int_{-1}^{1}x^{11}R_n(0,x)\,dx.\end{aligned}$$

Now,

$$\begin{aligned}\left|\int_{-1}^{1}x^{11}R_n(0,x)\,dx\right| &\le \int_{-1}^{1}\left|x^{11}\frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}\right|dx \le \int_{-1}^{1}\frac{|x^{n+12}|}{(n+1)!}dx\\ &= \frac{2}{(n+1)!}\int_{0}^{1}x^{n+12}\,dx = \frac{2}{(n+1)!}\left\{\frac{x^{n+13}}{n+13}\right\}_{0}^{1} = \frac{2}{(n+13)(n+1)!}.\end{aligned}$$

When $n=6$, this is less than $2.1\times 10^{-5}$. Hence, if we approximate the integral with the first three terms, namely, $\dfrac{2}{13}-\dfrac{2}{15\cdot 3!}+\dfrac{2}{17\cdot 5!}=0.132\,604$, then we can say that

$$0.132\,604-0.000\,021<\int_{-1}^{1}x^{11}\sin x\,dx<0.132\,604+0.000\,021,$$

that is, $0.132\,583<\displaystyle\int_{-1}^{1}x^{11}\sin x\,dx<0.132\,625$. To three decimals, the value of the integral is 0.133.

**14.** If we set $w=x^2$ and $dw=2x\,dx$, then $\displaystyle\int_0^{0.3}e^{-x^2}\,dx=\frac{1}{2}\int_0^{0.09}\frac{e^{-w}}{\sqrt{w}}dw$. Taylor's remainder formula applied to $e^{-w}$ gives

$$e^{-w}=1-w+\frac{w^2}{2!}-\frac{w^3}{3!}+\cdots+\frac{(-1)^n w^n}{n!}+R_n(0,w)$$

where $R_n(0,w)=\dfrac{d^{n+1}}{dw^{n+1}}(e^{-w})_{|w=w_n}\dfrac{w^{n+1}}{(n+1)!}=\dfrac{(-1)^{n+1}e^{-w_n}w^{n+1}}{(n+1)!}$. Consequently,

$$\begin{aligned}\int_0^{0.3}e^{-x^2}dx &= \frac{1}{2}\int_0^{0.09}\frac{1}{\sqrt{w}}\left[1-w+\frac{w^2}{2!}-\frac{w^3}{3!}+\cdots+\frac{(-1)^n w^n}{n!}+R_n(0,w)\right]dw\\ &= \frac{1}{2}\int_0^{0.09}\left[\frac{1}{\sqrt{w}}-\sqrt{w}+\frac{w^{3/2}}{2!}-\frac{w^{5/2}}{3!}+\cdots+\frac{(-1)^n w^{n-1/2}}{n!}+\frac{1}{\sqrt{w}}R_n(0,w)\right]dw\\ &= \frac{1}{2}\left\{2\sqrt{w}-\frac{2w^{3/2}}{3}+\frac{2w^{5/2}}{5\cdot 2!}-\frac{2w^{7/2}}{7\cdot 3!}+\cdots+\frac{2(-1)^n w^{n+1/2}}{(2n+1)n!}\right\}_0^{0.09}+\frac{1}{2}\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\\ &= \sqrt{0.09}-\frac{(0.09)^{3/2}}{3}+\frac{(0.09)^{5/2}}{5\cdot 2!}-\frac{(0.09)^{7/2}}{7\cdot 3!}+\cdots+\frac{(-1)^n(0.09)^{n+1/2}}{(2n+1)n!}+\frac{1}{2}\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw.\end{aligned}$$

Now,

$$\frac{1}{2}\left|\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\right|\le\frac{1}{2}\int_0^{0.09}\frac{1}{\sqrt{w}}\left|\frac{(-1)^{n+1}e^{-w_n}w^{n+1}}{(n+1)!}\right|dw=\frac{1}{2(n+1)!}\int_0^{0.09}e^{-w_n}w^{n+1/2}dw.$$

Since $0<w_n<w<0.09$, we can say $e^{-w_n}\le 1$. Thus,

$$\frac{1}{2}\left|\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\right|\le\frac{1}{2(n+1)!}\left\{\frac{2w^{n+3/2}}{2n+3}\right\}_0^{0.09}=\frac{(0.09)^{n+3/2}}{(2n+3)(n+1)!}.$$

When $n = 2$, this is less than $3.0 \times 10^{-6}$. Hence, if we approximate the integral with the first three terms, namely, $\sqrt{0.09} - \dfrac{(0.09)^{3/2}}{3} + \dfrac{(0.09)^{5/2}}{5 \cdot 2!} = 0.291\,243$, then we can say that

$$0.291\,243 - 0.000\,003 < \int_0^{0.3} e^{-x^2}\,dx < 0.291\,243 + 0.000\,003,$$

that is, $0.291\,240 < \displaystyle\int_0^{0.3} e^{-x^2}\,dx < 0.291\,246$. To three decimals, the value of the integral is 0.291.

**15.** Using the result of Example 10.24, $\displaystyle\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0}\left(1 + \frac{x^2}{3} + \frac{2x^4}{15} + \cdots\right) = 1.$

**16.** $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right] = \lim_{x\to 0}\left(\frac{1}{2!} - \frac{x^2}{4!} + \cdots\right) = \frac{1}{2}$

**17.** $\displaystyle\lim_{x\to 0} \frac{(1-\cos x)^2}{3x^4} = \lim_{x\to 0} \frac{1}{3x^4}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right]^2 = \lim_{x\to 0} \frac{1}{3x^4}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2$

$$= \lim_{x\to 0} \frac{1}{3x^4}\left(\frac{x^4}{4} - \frac{x^6}{24} + \cdots\right) = \lim_{x\to 0}\left(\frac{1}{12} - \frac{x^2}{72} + \cdots\right) = \frac{1}{12}.$$

**18.** $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{x}\left[\left(1 + \frac{x}{2} - \frac{(1/2)(-1/2)}{2!}x^2 + \cdots\right) - 1\right] = \lim_{x\to 0}\left[\frac{1}{2} + \frac{x}{8} + \cdots\right] = \frac{1}{2}$

**19.** $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} x\left(\frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \cdots\right) = \lim_{x\to\infty}\left(1 - \frac{1}{3!x^2} + \frac{1}{5!x^4} - \cdots\right) = 1$

**20.** $\displaystyle\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + \left[1 - 2x + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \cdots\right]}{1 - \left[1 - 2x + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \cdots\right]} = \frac{2 - 2x + 2x^2 - \dfrac{4x^3}{3} + \cdots}{2x - 2x^2 + \dfrac{4x^3}{3} + \cdots}$

Long division gives $\displaystyle\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{x} + \frac{x}{3} + \cdots$.
Thus, $\displaystyle\lim_{x\to 0}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x}\right) = \lim_{x\to 0}\left[\left(\frac{1}{x} + \frac{x}{3} + \cdots\right) - \frac{1}{x}\right] = 0.$

**21.** Taylor's remainder formula for $\sin(x/3)$ gives

$$\sin(x/3) = \frac{x}{3} - \frac{x^3}{3^3 \cdot 3!} + \frac{x^5}{3^5 \cdot 5!} - \frac{x^7}{3^7 \cdot 7!} + \cdots + \frac{d^n}{dx^n}[\sin(x/3)]_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}[\sin(x/3)]}{dx^{n+1}}_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!}$ and $z_n$ is between 0 and $x$. Since the $(n+1)^{\text{th}}$ derivative of $\sin(x/3)$ is $\pm 3^{-n-1}\sin(x/3)$ or $\pm 3^{-n-1}\cos(x/3)$, and $|x| \le 4$, it follows that

$$|R_n(0,x)| \le \frac{|x|^{n+1}}{3^{n+1}(n+1)!} \le \frac{4^{n+1}}{3^{n+1}(n+1)!}.$$

The smallest integer for which this is less than $10^{-3}$ is $n = 7$. Thus, the series should be truncated after $x^7/(3^7 \cdot 7!)$.

**22.** We set $u = x^3$ and consider the function $f(u) = 1/\sqrt{1+u}$ on the interval $0 < u < 1/8$. Since the $n^{\text{th}}$ derivative of $f(u)$ is $f^{(n)}(u) = \dfrac{(-1)^n[1 \cdot 3 \cdot 5 \cdots (2n-1)]}{2^n(1+u)^{n+1/2}}$, Taylor's remainder formula gives

$$f(u) = 1 - \frac{u}{2} + \frac{3u^2}{8} - \frac{5u^3}{16} + \cdots + \frac{(-1)^n[1 \cdot 3 \cdot 5 \cdots (2n-1)]}{2^n\, n!}u^n + R_n(0,u),$$

where $R_n(0,u) = \dfrac{f^{(n+1)}(z_n)}{(n+1)!}u^{n+1}$, and $0 < z_n < u$. Since $0 < u < 1/8$, we can say that

$$|R_n(0,u)| = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}|1+z_n|^{n+3/2}(n+1)!}|u|^{n+1} < \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}|1+0|^{n+3/2}(n+1)!}|u|^{n+1}$$
$$< \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}(n+1)!}\left(\frac{1}{8}\right)^{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{4n+4}(n+1)!}.$$

The smallest integer for which this is less than $10^{-4}$ is $n=3$. Thus, we should approximate $1/\sqrt{1+u}$ with $1-u/2+3u^2/8-5u^3/16$, or approximate $1/\sqrt{1+x^3}$ with

$$1-\frac{x^3}{2}+\frac{3x^6}{8}-\frac{5x^9}{16}.$$

**23.** Since the $n^{\text{th}}$ derivative of $f(x)=\ln(1-x)$ is $f^{(n)}(x)=-(n-1)!/(1-x)^n$, Taylor's remainder formula gives

$$f(x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots-\frac{x^n}{n}+R_n(0,x),$$

where $R_n(0,x)=\dfrac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1} = \dfrac{-n!\,x^{n+1}}{(n+1)!(1-z_n)^{n+1}} = \dfrac{-x^{n+1}}{(n+1)(1-z_n)^{n+1}}$, and $z_n$ is between 0 and $x$. Since $|x|<1/3$, we can say that

$$|R_n(0,x)| = \frac{|x|^{n+1}}{(n+1)|1-z_n|^{n+1}} < \frac{|x|^{n+1}}{(n+1)|1-1/3|^{n+1}} < \frac{(1/3)^{n+1}}{(n+1)(2/3)^{n+1}} = \frac{1}{(n+1)2^{n+1}}.$$

The smallest integer for which this is less than $10^{-2}$ is $n=4$. Thus, we should approximate $\ln(1-x)$ with $-x-x^2/2-x^3/3-x^4/4$.

**24.** Taylor's remainder formula for $\cos^2 x=(1+\cos 2x)/2$ gives

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1+\left(1-\frac{2^2x^2}{2!}+\frac{2^4x^4}{4!}-\cdots+\frac{f^{(n)}(0)}{n!}x^n+R_n(0,x)\right)\right]$$
$$= 1-x^2+\frac{x^4}{3}-\cdots+\frac{f^{(n)}(0)}{2n!}x^n+\frac{1}{2}R_n(0,x),$$

where $R_n(0,x)=\dfrac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}$ and $z_n$ is between 0 and $x$. Since the $(n+1)^{\text{th}}$ derivative of $f(x)$ is $\pm 2^{n+1}\sin 2x$ or $\pm 2^{n+1}\cos 2x$, and $|x|\le 0.1$, it follows that

$$\frac{1}{2}|R_n(0,x)| \le \frac{2^{n+1}|x|^{n+1}}{2(n+1)!} < \frac{2^n}{(n+1)!10^{n+1}}.$$

The smallest integer for which this is less than $10^{-3}$ is $n=2$. Thus, the function should be approximated by $1-x^2$.

**25.** If we substitute $y=f(x)=\sum_{n=0}^{\infty}a_nx^n$ into the differential equation,

$$0 = -4+\sum_{n=0}^{\infty}3a_nx^n+\sum_{n=0}^{\infty}na_nx^{n-1} = -4+\sum_{n=0}^{\infty}3a_nx^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$
$$= (-4+3a_0+a_1)+\sum_{n=1}^{\infty}[3a_n+(n+1)a_{n+1}]x^n.$$

When we equate coefficients to zero:

$$-4+3a_0+a_1=0 \quad \text{and} \quad 3a_n+(n+1)a_{n+1}=0, \quad n\ge 1.$$

The first implies that $a_1=4-3a_0$ and the second gives the recursive formula $a_{n+1}=\dfrac{-3a_n}{(n+1)}$, $n\ge 1$. Iteration leads to

$$a_2 = -\frac{3a_1}{2} = \frac{-3(4-3a_0)}{2}, \quad a_3 = -\frac{3a_2}{3} = \frac{3^2(4-3a_0)}{3!}, \quad a_4 = -\frac{3a_3}{4} = -\frac{3^3(4-3a_0)}{4!}, \quad \cdots .$$

Thus,

$$\begin{aligned} y = f(x) &= a_0 + (4-3a_0)x - \frac{3(4-3a_0)}{2!}x^2 + \frac{3^2(4-3a_0)}{3!}x^3 + \cdots \\ &= a_0 + \frac{(4-3a_0)}{3}\left[3x - \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots\right] \\ &= a_0 + \frac{(4-3a_0)}{3}(1-e^{-3x}) = \frac{4}{3} + \frac{(3a_0-4)}{3}e^{-3x}. \end{aligned}$$

**26.** If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty}[(n+1)n a_{n+1} + n a_n]x^{n-1}. \end{aligned}$$

When we equate coefficients to zero:

$$(n+1)n a_{n+1} + n a_n = 0 \quad \Longrightarrow \quad a_{n+1} = -\frac{a_n}{n+1}, \quad n \geq 1.$$

This recursive definition implies that

$$a_2 = -\frac{a_1}{2}, \quad a_3 = -\frac{a_2}{3} = \frac{a_1}{3!}, \quad a_4 = -\frac{a_3}{4} = -\frac{a_1}{4!}, \quad \cdots .$$

Thus, $\quad y = f(x) = a_0 + a_1\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots\right) = a_0 + a_1 \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}x^n$

$$= a_0 - a_1(e^{-x} - 1) = (a_0 + a_1) - a_1 e^{-x}.$$

**27.** If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$0 = -3x - \sum_{n=0}^{\infty} 4a_n x^n + \sum_{n=0}^{\infty} n a_n x^n = -4a_0 + (-3 - 4a_1 + a_1)x + \sum_{n=2}^{\infty}(n-4)a_n x^n.$$

When we equate coefficients to zero:

$$a_0 = 0, \qquad -3 - 3a_1 = 0, \qquad (n-4)a_n = 0, \quad n \geq 2.$$

These imply that $a_1 = -1$, $a_4$ is undetermined, and all other coefficients vanish. Thus, $y = f(x) = -x + a_4 x^4$.

**28.** If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= 4x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \\ &= (2a_1 + a_0) + \sum_{n=2}^{\infty}[(4n^2 - 4n + 2n)a_n + a_{n-1}]x^{n-1}. \end{aligned}$$

We now equate coefficients of powers of $x$ to zero. From the coefficient of $x^0$ we obtain $2a_1 + a_0 = 0$ which implies that $a_1 = -a_0/2$. From the remaining coefficients, we obtain

$$(4n^2 - 2n)a_n + a_{n-1} = 0 \quad \Longrightarrow \quad a_n = \frac{-a_{n-1}}{2n(2n-1)}, \quad n \geq 2.$$

When we iterate this recursive definition:

$$a_2 = \frac{-a_1}{4 \cdot 3} = \frac{a_0}{4!}, \quad a_3 = \frac{-a_2}{6 \cdot 5} = -\frac{a_0}{6!}, \quad \cdots$$

The solution is therefore $\quad y = f(x) = a_0 \left(1 - \dfrac{x}{2} + \dfrac{x^2}{4!} - \dfrac{x^3}{6!} + \cdots \right) = a_0 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n.$

**29.** If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n. \end{aligned}$$

When we equate coefficients of powers of $x$ to zero, we obtain the recursive formula

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Iteration gives

$$a_2 = -\frac{a_0}{2!}, \quad a_4 = \frac{a_0}{4!}, \quad a_6 = -\frac{a_0}{6!}, \quad \ldots, \quad \text{and} \quad a_3 = -\frac{a_1}{3!}, \quad a_5 = \frac{a_1}{5!}, \quad a_7 = -\frac{a_1}{7!}, \quad \ldots.$$

The solution is therefore

$$y = f(x) = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) = a_0 \cos x + a_1 \sin x.$$

**30.** If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= x \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = a_0 + \sum_{n=2}^{\infty} [n(n-1)a_n + a_{n-1}] x^{n-1}. \end{aligned}$$

We now equate coefficients of powers of $x$ to zero. From the coefficient of $x^0$ we obtain $a_0 = 0$. From the remaining coefficients, we obtain

$$n(n-1)a_n + a_{n-1} = 0 \quad \Longrightarrow \quad a_n = \frac{-a_{n-1}}{n(n-1)}, \quad n \geq 2.$$

When we iterate this recursive definition:

$$a_2 = \frac{-a_1}{2 \cdot 1}, \quad a_3 = \frac{-a_2}{3 \cdot 2} = \frac{a_1}{3!\,2!}, \quad a_4 = \frac{-a_3}{4 \cdot 3} = \frac{-a_1}{4!\,3!}, \quad \cdots.$$

The solution is therefore

$$y = f(x) = a_1 \left(x - \frac{x^2}{2 \cdot 1} + \frac{x^3}{3! \cdot 2!} - \frac{x^4}{4! \cdot 3!} + \cdots \right) = a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!(n-1)!} x^n.$$

**31.** According to Exercise 23, Taylor's remainder formula for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \cdots + R_n(0,x), \quad \text{where } R_n(0,x) = \frac{-x^{n+1}}{(n+1)(1-z_n)^{n+1}},$$

and $z_n$ is between 0 and $x$. The maximum error when only the first term is used is $R_1(0,x) = \dfrac{-x^2}{2(1-z_1)^2}$. If we set $x = 0.000\,000\,000\,1$, then $z_1 < 0.000\,000\,000\,1$, and we can say that

$$|R_1(0, 0.000\,000\,000\,1)| < \frac{(0.000\,000\,000\,1)^2}{2(0.999\,999\,999\,9)^2} < 3.4 \times 10^{-21}.$$

Hence, $\ln(0.999\,999\,999\,9) = -10^{-10}$, and this is definitely accurate to more than 15 decimal places.

**32.** $$K = c^2(m - m_0) = c^2 m_0 \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$
$$= c^2 m_0 \left\{\left[1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{(-1/2)(-3/2)}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(-\frac{v^2}{c^2}\right)^3 + \cdots\right] - 1\right\}$$
$$= c^2 m_0 \left\{\frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right\} = \frac{1}{2}m_0 v^2 + m_0 c^2\left(\frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

**33.** Using the binomial expansion,

$$\frac{P_0}{P} = 1 + \left(\frac{k}{k-1}\right)\left(\frac{k-1}{2}\right)M^2 + \cdots = 1 + \frac{kM^2}{2} + \cdots .$$

**34.** When we expand $P_s/P_0$ with the binomial expansion,

$$\begin{aligned}\frac{P_s}{P_0} &= 1 + \left(\frac{k}{k-1}\right)\left(\frac{k-1}{2}\right)M_0^2 + \frac{1}{2}\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1} - 1\right)\left(\frac{k-1}{2}\right)^2 M_0^4 \\ &\quad + \frac{1}{3!}\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1} - 1\right)\left(\frac{k}{k-1} - 2\right)\left(\frac{k-1}{2}\right)^3 M_0^6 + \cdots \\ &= 1 + \frac{k}{2}M_0^2 + \frac{k}{8}M_0^4 + \frac{k(2-k)}{48}M_0^6 + \cdots \\ &= 1 + \frac{1}{2}M_0^2\left(\frac{\rho_0 c_0^2}{P_0}\right) + \frac{1}{8}M_0^4\left(\frac{\rho_0 c_0^2}{P_0}\right) + \frac{1}{48}M_0^6(2-k)\left(\frac{\rho_0 c_0^2}{P_0}\right) + \cdots .\end{aligned}$$

Multiplication by $P_0$, and replacement of $M_0^2$ by $V_0^2/c_0^2$ in the last three terms gives

$$\begin{aligned}P_s &= P_0 + \frac{1}{2}\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right) + \frac{1}{8}\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right)M_0^2 + \frac{1}{48}(2-k)\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right)M_0^4 + \cdots \\ &= P_0 + \frac{1}{2}\rho_0 V_0^2 + \frac{1}{8}\rho_0 V_0^2 M_0^2 + \frac{1}{48}(2-k)\rho_0 V_0^2 M_0^4 + \cdots \\ &= P_0 + \frac{1}{2}\rho_0 V_0^2\left[1 + \frac{M_0^2}{4} + \left(\frac{2-k}{24}\right)M_0^4 + \cdots\right].\end{aligned}$$

**35.** (a) Using formula 9.3, the length of the ellipse is four times that in the first quadrant,

$$L = 4\int_0^{\pi/2}\sqrt{(-a\sin t)^2 + (b\cos t)^2}\,dt = 4b\int_0^{\pi/2}\sqrt{\frac{a^2}{b^2}\sin^2 t + (1 - \sin^2 t)}\,dt = 4b\int_0^{\pi/2}\sqrt{1 - k^2\sin^2 t}\,dt,$$

where $k^2 = 1 - a^2/b^2$.
(b) If we expand the integrand with the binomial expansion 10.33b, and integrate term-by-term,

$$\begin{aligned}
L &= 4b\int_0^{\pi/2}\left[1+\frac{1}{2}(-k^2\sin^2 t)+\frac{(1/2)(-1/2)}{2}(-k^2\sin^2 t)^2+\cdots\right]dt\\
&= 4b\int_0^{\pi/2}\left[1-\frac{k^2}{2}\left(\frac{1-\cos 2t}{2}\right)-\frac{k^4}{8}\left(\frac{1-\cos 2t}{2}\right)^2+\cdots\right]dt\\
&= 4b\int_0^{\pi/2}\left[1-\frac{k^2}{4}(1-\cos 2t)-\frac{k^4}{32}\left(1-2\cos 2t+\frac{1+\cos 4t}{2}\right)+\cdots\right]dt\\
&= 4b\left\{t-\frac{k^2}{4}\left(t-\frac{\sin 2t}{2}\right)-\frac{k^4}{32}\left(\frac{3t}{2}-\sin 2t+\frac{\sin 4t}{8}\right)+\cdots\right\}_0^{\pi/2}\\
&= 4b\left[\frac{\pi}{2}-\frac{k^2}{4}\left(\frac{\pi}{2}\right)-\frac{k^4}{32}\left(\frac{3\pi}{4}\right)+\cdots\right]\\
&= 2\pi b\left(1-\frac{k^2}{4}-\frac{3k^4}{64}+\cdots\right).
\end{aligned}$$

**36.** (a) If we substitute $e^{-\beta^2/(4\alpha x)}=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha x}\right)^n$, we obtain

$$W(\alpha,\beta)=\int_1^{\infty}\frac{1}{x}e^{-\alpha x}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha x}\right)^n dx=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha}\right)^n\int_1^{\infty}\frac{e^{-\alpha x}}{x^{n+1}}dx=\sum_{n=0}^{\infty}\frac{(-1)^n\beta^{2n}}{4^n\alpha^n n!}E_{n+1}(\alpha).$$

(b) We use integration by parts with $u=e^{-\alpha x}$ and $dv=\dfrac{1}{x^{n+1}}dx$,

$$\begin{aligned}
E_{n+1}(\alpha) &= \int_1^{\infty}\frac{e^{-\alpha x}}{x^{n+1}}dx=\left\{-\frac{e^{-\alpha x}}{nx^n}\right\}_1^{\infty}-\int_1^{\infty}-\frac{1}{nx^n}(-\alpha)e^{-\alpha x}\,dx=\frac{e^{-\alpha}}{n}-\frac{\alpha}{n}\int_1^{\infty}\frac{e^{-\alpha x}}{x^n}dx\\
&= \frac{1}{n}[e^{-\alpha}-\alpha E_n(\alpha)].
\end{aligned}$$

**37.** If we substitute the Maclaurin series for $e^{ch/(\lambda kT)}$,

$$\Psi(\lambda)=\frac{8\pi ch\lambda^{-5}}{\left(1+\dfrac{ch}{\lambda kT}+\dfrac{c^2h^2}{2\lambda^2k^2T^2}+\cdots\right)-1}=\frac{8\pi ch}{\lambda^5\left(\dfrac{ch}{\lambda kT}+\dfrac{c^2h^2}{2\lambda^2k^2T^2}+\cdots\right)}=\frac{8\pi ch}{\dfrac{ch}{kT}\lambda^4+\dfrac{c^2h^2}{2k^2T^2}\lambda^3+\cdots}.$$

If we long divide the denominator into the numerator, the result is

$$\Psi(\lambda)=\frac{8\pi kT}{\lambda^4}+\text{ terms in }\lambda^{-5},\ \lambda^{-6},\text{ etc..}$$

Thus, for large $\lambda$, $\Psi(\lambda)$ can be approximated by $8\pi kT/\lambda^4$.

**38.** (a) We write $E=\dfrac{q}{4\pi\epsilon_0x^2\left(1-\dfrac{d}{2x}\right)^2}-\dfrac{q}{4\pi\epsilon_0x^2\left(1+\dfrac{d}{2x}\right)^2}=\dfrac{q}{4\pi\epsilon_0x^2}\left[\left(1-\dfrac{d}{2x}\right)^{-2}-\left(1+\dfrac{d}{2x}\right)^{-2}\right].$

(b) If we expand each term with the binomial expansion 10.33b,

$$E=\frac{q}{4\pi\epsilon_0x^2}\left\{\left[1-2\left(-\frac{d}{2x}\right)+\cdots\right]-\left[1-2\left(\frac{d}{2x}\right)+\cdots\right]\right\}.$$

When $d$ is very much less than $x$, we omit higher order terms in $d/x$, and write

$$E\approx\frac{q}{4\pi\epsilon_0x^2}\left(1+\frac{d}{x}-1+\frac{d}{x}\right)=\frac{qd}{2\pi\epsilon_0x^3}.$$

**39.** The cross-sectional area of the liquid is the area of the sector less the area of the triangle above it,

$$A = \frac{1}{2}R^2\theta - 2\left(\frac{1}{2}\right)\left(R\sin\frac{\theta}{2}\right)\left(R\cos\frac{\theta}{2}\right) = \frac{R^2}{2}(\theta - \sin\theta).$$

Since $d = 2R\sin\dfrac{\theta}{2}$ and $h = R - R\cos\dfrac{\theta}{2}$,

$$hd = 2R\sin\frac{\theta}{2}\left(R - R\cos\frac{\theta}{2}\right) = R^2\left(2\sin\frac{\theta}{2} - \sin\theta\right).$$

The required ratio is

$$\frac{A}{hd} = \frac{\dfrac{R^2}{2}(\theta - \sin\theta)}{R^2\left(2\sin\dfrac{\theta}{2} - \sin\theta\right)} = \frac{\theta - \sin\theta}{2\left(2\sin\dfrac{\theta}{2} - \sin\theta\right)}.$$

If we expand the sine functions in their Maclaurin series

$$\begin{aligned}\frac{A}{hd} &= \frac{\theta - \left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots\right)}{2\left[2\left(\dfrac{\theta}{2} - \dfrac{(\theta/2)^3}{3!} + \dfrac{(\theta/2)^5}{5!} - \cdots\right) - \left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots\right)\right]}\\ &= \frac{\dfrac{\theta^3}{3!} - \dfrac{\theta^5}{5!} + \cdots}{\dfrac{\theta^3}{4} - \dfrac{\theta^5}{64} + \cdots} \qquad \text{(and by long division)}\\ &= \frac{2}{3} + \frac{\theta^2}{120} + \cdots .\end{aligned}$$

For small $\theta$, we can use the approximation $\quad \dfrac{A}{hd} \approx \dfrac{2}{3} + \dfrac{\theta^2}{120}.$

**EXERCISES 10.8**

**1.** True If a sequence satisfies 10.35a, then it satisfies 10.35b; that is, every increasing sequence is nondecreasing.

**2.** False The sequence $\{n\}$ is increasing but has no upper bound.

**3.** True The first term of an increasing sequence is a lower bound.

**4.** False The sequence $\{-n\}$ is decreasing but has no lower bound.

**5.** False The sequence $\{n\}$ is increasing with lower bound 1, but it does not have a limit.

**6.** True An increasing sequence has a lower bound. If it also has an upper bound, then it has a limit according to Theorem 10.7.

**7.** False The sequence $\{(-1)^n\}$ does not converge, but its terms are all $\pm 1$.

**8.** True For a sequence to be increasing and decreasing, its terms would have to satisfy $c_{n+1} > c_n$ and $c_{n+1} < c_n$ for all $n$. This is impossible.

**9.** True The sequence $\{1\}$ is an example.

**10.** True This is part of the corollary to Theorem 10.7.

**11.** False The sequence $\{(-1)^n/n\}$ is bounded and has limit 0, but it is not monotonic.

**12.** False The sequence $\{(-1)^n/n\}$ is bounded, not monotonic, and it has limit 0.