MATH2132 Test1 **February 9, 2016** **60 minutes**

Student Name - **Student Number -**

Values

5 **1.** Find the limit of the sequence of functions

$$\left\{\frac{n^3x^4+(3n^2-1)x+4}{n^3x^2+13nx+5}\right\}$$

on the interval $0 \le x \le 2$, if it exists.

$$\lim_{n\to\infty}\frac{n^3x^4+(3n^2-1)x+4}{n^3x^2+13nx+5}=\lim_{n\to\infty}\frac{x^4+\left(\dfrac{3}{n}-\dfrac{1}{n^3}\right)x+\dfrac{4}{n^3}}{x^2+\dfrac{13x}{n^2}+\dfrac{5}{n^3}}=\frac{x^4}{x^2}=x^2,$$

provided $x \neq 0$. When $x = 0$ the sequence of functions is $\left\{\dfrac{4}{5}\right\}$, with limit $4/5$. Thus,

$$\lim_{n\to\infty}\frac{n^3x^4+(3n^2-1)x+4}{n^3x^2+13nx+5}=\begin{cases}x^2, & x\neq 0\\ 4/5, & x=0.\end{cases}$$

8 **2.** Find the interval of convergence for the power series

$$\sum_{n=4}^{\infty} \frac{1}{(n+1)3^n}(x+4)^{2n+3}.$$

When we set $y = (x+4)^2$, the series becomes

$$\sum_{n=4}^{\infty} \frac{1}{(n+1)3^n}(x+4)^{2n+3} = (x+4)^3 \sum_{n=4}^{\infty} \frac{1}{(n+1)3^n} y^n.$$

Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{1}{(n+1)3^n}}{\dfrac{1}{(n+2)3^{n+1}}} \right| = 3.$$

Thus, $R_x = \sqrt{3}$, and the open interval of convergence is

$$-\sqrt{3} < x+4 < \sqrt{3} \quad \Longrightarrow \quad -4-\sqrt{3} < x < -4+\sqrt{3}.$$

At the ends $x = -4 \pm \sqrt{3}$, the series becomes

$$\sum_{n=4}^{\infty} \frac{1}{(n+1)3^n}(\pm\sqrt{3})^{2n+3} = \sum_{n=4}^{\infty} \frac{\pm 3\sqrt{3}}{n+1} = \pm 3\sqrt{3} \sum_{n=4}^{\infty} \frac{1}{n+1}.$$

Since this is the harmonic series less the first four terms, it diverges. The interval of convergence is therefore $-4-\sqrt{3} < x < -4+\sqrt{3}$.

6 **3.** Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{(2n)!}(x-1)^{2n}$. Include its interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{(2n)!}(x-1)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}[\sqrt{3}(x-1)]^{2n} - 1 = \cos\sqrt{3}(x-1) - 1.$$

It converges for all x.

10 **4.** Determine whether the following series converge or diverge. Justify your answers. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{n^2+1}{3n^2+2}\right)^3$ (b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}2^n}{3^{2n}}$

(a) Since $\displaystyle\lim_{n\to\infty}\left(\frac{n^2+1}{3n^2+2}\right)^3=\frac{1}{27}\neq 0$, the series diverges by the n^{th}-term test.

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}2^n}{3^{2n}}=-\sum_{n=2}^{\infty}\left(-\frac{2}{9}\right)^n.$

The series is geometric with common ratio $-2/9$. It therefore converges with sum

$$\sum_{n=2}^{\infty}\frac{(-1)^{n+1}2^n}{3^{2n}}=-\frac{(-2/9)^2}{1+2/9}=-\frac{4}{99}.$$

11 **5.** Find the Taylor series about $x = 3$ (or, using $c = 3$) for the function

$$f(x) = \frac{(x-3)^2}{(x+2)^2}.$$

You must use a method that guarantees that the series converges to the function. Write the series in sigma notation, simplified as much as possible. Include the radius of convergence of the series.

$\boxed{x-3}$

Method 1 (using differentiation):

$$\frac{1}{x+2} = \frac{1}{5+(x-3)} = \frac{1/5}{1+(x-3)/5} = 1/5\sum_{n=0}^{\infty}\left(-\frac{x-3}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-3)^n,$$

valid for $\left|-\frac{x-3}{5}\right| < 1$, or $|x-3| < 5$. Differentiation gives

$$-\frac{1}{(x+2)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{5^{n+1}}(x-3)^{n-1}.$$

Thus,

$$\frac{(x-3)^2}{(x+2)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{5^{n+1}}(x-3)^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n(n-1)}{5^n}(x-3)^n = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{5^n}(x-3)^n.$$

The radius of convergence is $R = 5$.

Method 2(using binomial expansion):

$$\frac{1}{(x+2)^2} = \frac{1}{[5+(x-3)]^2} = \frac{1}{25\left(1+\frac{x-3}{5}\right)^2} = \frac{1}{25}\left(1+\frac{x-3}{5}\right)^{-2}$$

$$= \frac{1}{25}\left[1+(-2)\left(\frac{x-3}{5}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x-3}{5}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x-3}{5}\right)^3+\cdots\right]$$

$$= \frac{1}{5^2}-\frac{2}{5^3}(x-3)+\frac{3}{5^4}(x-3)^2-\frac{4}{5^5}(x-3)^3+\cdots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{5^{n+2}}(x-3)^n, \qquad \text{valid for } \left|\frac{x-3}{5}\right| < 1 \quad \Longrightarrow |x-3| < 5.$$

Thus,

$$\frac{(x-3)^2}{(x+2)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{5^{n+2}}(x-3)^{n+2} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{5^n}(x-3)^n.$$

The radius of convergence is $R = 5$.