MATH2132 Test1 **October 14, 2021** **60 minutes**

Values

8 **1.** Find the interval of convergence and the sum of the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2^n}(3x-1)^n.$$

It is not necessary to simplify the expression for the sum.

Writing the series in the form

$$\sum_{n=2}^{\infty} -\left[-\frac{1}{2}(3x-1)\right]^n$$

shows that it is geometric. Its sum is

$$\frac{-\frac{1}{4}(3x-1)^2}{1+\frac{1}{2}(3x-1)},$$

and the interval of convergence is

$$\left|-\frac{1}{2}(3x-1)\right| < 1 \quad \Longrightarrow \quad -2 < 3x-1 < 2 \quad \Longrightarrow \quad -1 < 3x < 3 \quad \Longrightarrow \quad -\frac{1}{3} < x < 1.$$

6 **2.** Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n a^n}{n^2}, \qquad a \geq 2 \text{ a constant},$$

converges or diverges. Justify all conclusions.

Consider the sequence $\left\{\dfrac{a^n}{n^2}\right\}$. To determine its limit as $n \to \infty$, we use L'Hospital's rule on the limit

$$\lim_{x\to\infty} \frac{a^x}{x^2} = \lim_{x\to\infty} \frac{a^x(\ln a)}{2x} = \lim_{x\to\infty} \frac{a^x(\ln a)^2}{2} = \infty.$$

Consequently, $\displaystyle\lim_{n\to\infty} \frac{(-1)^n a^n}{n^2}$ does not exist. The series therefore diverges by the n^{th}-term test.

14 **3.** Find the Taylor series about $x = 1$ for the function

$$\frac{1}{\sqrt{5x-1}}.$$

Write your answer in sigma notation simplified as much as possible. You must use a method that guarantees that the series converges to the function. What is the radius of convergence of the series?

$\boxed{x-1}$

$$\begin{aligned}
\frac{1}{\sqrt{5x-1}} &= \frac{1}{\sqrt{5(x-1)+4}} = \frac{1}{2}\left[1+\frac{5}{4}(x-1)\right]^{-1/2} \\
&= \frac{1}{2}\left\{1+(-1/2)\left[\frac{5}{4}(x-1)\right] + \frac{(-1/2)(-3/2)}{2!}\left[\frac{5}{4}(x-1)\right]^2 \right. \\
&\qquad \left. + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left[\frac{5}{4}(x-1)\right]^3 + \cdots\right\} \\
&= \frac{1}{2}\left\{1-\frac{5}{2\cdot 4}(x-1)+\frac{(1\cdot 3)5^2}{2^2\,2!\,4^2}(x-1)^2-\frac{(1\cdot 3\cdot 5)5^3}{2^3\,3!\,4^3}(x-1)^3+\cdots\right\} \\
&= \frac{1}{2}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n 5^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n\,n!\,4^n}(x-1)^n\right\} \\
&= \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n 5^n (2n)!}{2^n\,2^{2n}\,2^n\,n!\,n!}(x-1)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 5^n (2n)!}{2^{4n+1}(n!)^2}(x-1)^n
\end{aligned}$$

This is valid for $\left|\frac{5}{4}(x-1)\right| < 1 \quad \Longrightarrow \quad |x-1| < \frac{4}{5}$. The radius of convergence is $4/5$.

12 **4.** If the first four terms in the Maclaurin series for the function

$$f(x) = \frac{1}{1+4x}$$

are used as an approximation to the function on the interval $0 \le x \le 0.1$, what is the maximum possible error? Justify all conclusions.

The Maclaurin series for the function is

$$f(x) = \sum_{n=0}^{\infty} (-4x)^n = 1 - 4x + 16x^2 - 64x^3 + 256x^4 + \cdots$$

When values of x in the interval $0 \le x \le 0.1$ are substituted into the series, the series is alternating. Absolute values of terms in the series are $\{4^n x^n\}$. The largest value of x is 0.1, and for this value, terms are $\{4^n/10^n\}$ which decrease and approach 0. Smaller values of x do likewise. The series therefore passes the alternating series test, and when the series is truncated after the fourth term, the maximum error is the next term. The maximum error is therfore

$$256x^4 \le 256(0.1)^4 = 0.0256.$$

Alternatively, when the series is truncated after the term in x^3, Taylor's remainder formula gives the error

$$|R_3(0,x)| = \left|\frac{f^{(4)}(z_3)x^4}{4!}\right| = \left|\frac{6144}{(1+4z_3)^5}\frac{x^4}{4!}\right| \le \frac{6144}{(1+0)^5}\frac{x^4}{4!} \le \frac{6144(0.1)^4}{4!} = 0.0256.$$