MATH2132 Test1 **October 13, 2022** **60 minutes**

Values

10 **1.** Find the interval of convergence for the series

$$\sum_{n=4}^{\infty} \frac{n^2 3^n}{n^2+1}(x+3)^{2n}.$$

Justify all conclusions.

When we set $y = (x+3)^2$, the series becomes $\displaystyle\sum_{n=4}^{\infty} \frac{n^2 3^n}{n^2+1} y^n$. The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{n^2 3^n}{n^2+1}}{\frac{(n+1)^2 3^{n+1}}{(n+1)^2+1}} \right| = \frac{1}{3}.$$

Hence, $R_x = 1/\sqrt{3}$, and the open interval of convergence is

$$-\frac{1}{\sqrt{3}} < x+3 < \frac{1}{\sqrt{3}} \quad \Longrightarrow \quad -3 - \frac{1}{\sqrt{3}} < x < -3 + \frac{1}{\sqrt{3}}.$$

At the end points, the series becomes

$$\sum_{n=4}^{\infty} \frac{n^2 3^n}{n^2+1}\left(\pm\frac{1}{\sqrt{3}}\right)^{2n} = \sum_{n=4}^{\infty} \frac{n^2}{n^2+1}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$, this series diverges by the n^{th}-term test. The interval of convergence is therefore $-3 - \dfrac{1}{\sqrt{3}} < x < -3 + \dfrac{1}{\sqrt{3}}$.

5 **2.** Determine whether the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n 2^{n+5}}{e^{n-1}}$$

converges or diverges. Justify your answer. If the series converges, find its sum.

Since we can write the series in the form $\displaystyle\sum_{n=3}^{\infty} \frac{2^5}{e^{-1}}\left(-\frac{2}{e}\right)^n$, we see that it is geometric with common ratio $-2/e$. Since this is between -1 and 1, the series converges, and has sum

$$\frac{\frac{(-1)^3 2^8}{e^2}}{1+2/e}.$$

5 **3.** Find the limit of the sequence of functions $\left\{\dfrac{n^2x^3+nx^2+5}{2n^2x^3+3}\right\}$ on the interval $-2 \le x \le 0$, if it exists. If the limit does not exist, explain why not.

$$\lim_{n\to\infty}\frac{n^2x^3+nx^2+5}{2n^2x^3+3} = \lim_{n\to\infty}\frac{x^3+x^2/n+5/n^2}{2x^3+3/n^2} = \frac{x^3}{2x^3} = \frac{1}{2},$$

provided $x \neq 0$. At $x = 0$, terms in the sequence are all equal to $5/3$ so that the limit at $x = 0$ is $5/3$. Thus,

$$\lim_{n\to\infty}\frac{n^2x^3+nx^2+5}{2n^2x^3+3} = \begin{cases} 1/2, & -2 \le x < 0 \\ 5/3, & x = 0. \end{cases}$$

5 **4.** You are given that the Taylor remainder about $x = 0$ for a function $f(x)$ on the interval $0 \le x \le 2$ is

$$R_n(0,x) = \frac{z_n}{(5-z_n)^n}\frac{x^{n+1}}{(n+1)!}.$$

Show that $\lim_{n\to\infty} R_n(0,x) = 0$. Explain your reasoning.

$$|R_n(0,x)| = \frac{|z_n|}{|5-z_n|^n}\frac{|x|^{n+1}}{(n+1)!}, \qquad \text{where } 0 < z_n < x \le 2.$$

Since the numerator of $\dfrac{|z_n|}{|5-z_n|^n}$ is largest when $|z_n|$ is largest, and the denominator $|5-z_n|^n$ is smallest when z_n is largest, we can say that

$$|R_n(0,x)| < \frac{2}{(5-2)^n}\frac{|x|^{n+1}}{(n+1)!} = 2\frac{|x/3|^n}{(n+1)!}.$$

This approaches 0 as $n \to \infty$. Hence,

$$\lim_{n\to\infty} R_n(0,x) = 0.$$

15 **5.** Find the Taylor series about $x = -2$ for the function

$$f(x) = \frac{x+2}{\sqrt{x+3}}.$$

Write your answer in sigma notation simplified as much as possible. You must use a method that guarantees that the series converges to the function. What is the radius of convergence of the series?

x+2

$$\begin{aligned}
\frac{1}{\sqrt{x+3}} &= \frac{1}{\sqrt{1+(x+2)}} = [1+(x+2)]^{-1/2} \\
&= 1 + (-1/2)(x+2) + \frac{(-1/2)(-3/2)}{2!}(x+2)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(x+2)^3 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n n!}(x+2)^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n)]}{2^n n![2\cdot 4\cdots(2n)]}(x+2)^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{n!\,2^{2n}\,n!}(x+2)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}(x+2)^n
\end{aligned}$$

Thus,

$$\frac{x+2}{\sqrt{x+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}(x+2)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{2^{2n-2}[(n-1)!]^2}(x+2)^n.$$

This is valid for $|x+2| < 1$ which implies that the radius of convergence is 1.