MATH2132 Test1 **October 12, 2023** **60 minutes**

Values

14 **1.** Find the interval of convergence for the series

$$\sum_{n=3}^{\infty} \frac{\sqrt{n}}{3^{n+1}}(x+1)^{2n}.$$

Justify all conclusions.

If we set $y=(x+1)^2$, the series becomes $\displaystyle\sum_{n=3}^{\infty} \frac{\sqrt{n}}{3^{n+1}} y^n$.

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{\sqrt{n}}{3^{n+1}}}{\frac{\sqrt{n+1}}{3^{n+2}}} \right| = 3.$$

Hence, $R_x = \sqrt{3}$, and the open interval of convergence is $-\sqrt{3} < x+1 < \sqrt{3}$, or $-1-\sqrt{3} < x < -1+\sqrt{3}$. At the end points, the series becomes

$$\sum_{n=3}^{\infty} \frac{\sqrt{n}}{3^{n+1}}(\pm\sqrt{3})^{2n} = \sum_{n=3}^{\infty} \frac{\sqrt{n}}{3}.$$

Since $\lim_{n\to\infty} \sqrt{n}/3 = \infty$, the series diverges (by the n^{th}-term test). The interval of convergence is therefore $-1-\sqrt{3} < x < -1+\sqrt{3}$

11 **2.** Find the interval of convergence and the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{n+3}}{n!}x^{2n+1}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{n+3}}{n!}x^{2n+1} = \sum_{n=1}^{\infty} \frac{-8x(-2x^2)^n}{n!} = -8x\left[\sum_{n=0}^{\infty} \frac{(-2x^2)^n}{n!} - 1\right]$$
$$= -8x(e^{-2x^2} - 1) = 8x(1 - e^{-2x^2}).$$

The series converges for

$$-\infty < -2x^2 < \infty \quad \Longrightarrow \quad \infty > 2x^2 > -\infty \quad \Longrightarrow \quad 2x^2 < \infty \quad \Longrightarrow \quad x^2 < \infty \quad \Longrightarrow \quad -\infty < x < \infty.$$

16 **3.** Find the Taylor series of the function

$$f(x) = \frac{x}{4x+3}$$

about $x = 2$. Write your answer in sigma notation simplified as much as possible. You must use a method that guarantees that the series converges to the function. What is the radius of convergence of the series?

x-2

Long division gives

$$\frac{x}{4x+3} = \frac{1}{4} - \frac{3/4}{4x+3} = \frac{1}{4} - \frac{3}{4}\left[\frac{1}{4(x-2)+11}\right] = \frac{1}{4} - \frac{3}{44}\left[\frac{1}{1+(4/11)(x-2)}\right]$$

$$= \frac{1}{4} - \frac{3}{44}\sum_{n=0}^{\infty}\left[-\frac{4}{11}(x-2)\right]^n$$

$$= \frac{1}{4} - \frac{3}{44}\sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{11^n}(x-2)^n$$

$$= \frac{1}{4} - \frac{3}{44} + \sum_{n=1}^{\infty}\frac{3(-1)^{n+1}4^{n-1}}{11^{n+1}}(x-2)^n$$

$$= \frac{2}{11} + \sum_{n=1}^{\infty}\frac{3(-1)^{n+1}4^{n-1}}{11^{n+1}}(x-2)^n.$$

Since the series is valid for

$$-1 < -\frac{4}{11}(x-2) < 1 \qquad \Longrightarrow \qquad -\frac{11}{4} < x-2 < \frac{11}{4},$$

the radius of convergence is 11/4.

6 **4** If $a > 1$ is a constant, determine whether the the sequence of functions

$$f_n(x) = \frac{a^n x^3 - 2ax + 1}{a^{n+1}x^2 + 3x + 10}$$

has a limit on the interval $0 \le x \le 14$.

When we divide all terme by a^n,

$$\lim_{n\to\infty} \frac{a^n x^3 - 2ax + 1}{a^{n+1}x^2 + 3x + 10} = \lim_{n\to\infty} \frac{x^3 - 2x/a^{n-1} + 1/a^n}{ax^2 + 3x/a^n + 10/a^n} = \frac{x^3}{ax^2} = \frac{x}{a},$$

provided $x \ne 0$. When $x = 0$,

$$f_n(0) = \frac{1}{10}, \qquad \text{and} \qquad \lim_{n\to\infty} f_n(0) = \frac{1}{10}.$$

Thus,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} x/a, & 0 < x \le 14 \\ 1/10, & x = 0. \end{cases}$$

3 **5.** Find, if possible, an example of a power series in powers of $x+5$ that has open interval of convergence $-20 < x < 11$. If it is not possible, explain why not.

This is not possible. The centre of the interval is $x = -4.5$. It should be $x = -5$ for a power series in $x + 5$.