MATH2132 Test1 **May 2014** **60 minutes**

8 **1.** Determine whether the sequence of functions

$$\{f_n(x)\} = \left\{\left(\frac{n}{n+1}\right)x + \left(\frac{n+1}{n}\right)^n x^n\right\}$$

has a limit on the interval $0 \le x \le 1$. Show your reasoning and all calculations.

Since $\lim_{n\to\infty} x^n = 0$ for $0 < x < 1$, it follows that

$$\lim_{n\to\infty} f_n(x) = x.$$

At $x = 1$, however,

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)(1) + \left(\frac{n+1}{n}\right)^n (1)^n\right] = 1 + e.$$

Thus,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} x, & 0 < x < 1, \\ 1+e, & x = 1. \end{cases}$$

10 **2.** Determine whether the series in parts (a) and (b) converge or diverge. If a series converges, find its sum simplified as much as possible. Justify your conclusions. You do not have to do part (c), but if you do, there is a 5 mark bonus.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{2^{2n}}{5^{n+2}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+4}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+n}$

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{2^{2n}}{5^{n+2}} = \sum_{n=3}^{\infty} \frac{1}{25}\left(\frac{4}{5}\right)^n$ is a geometric series with common ratio $r = 4/5$. The series therefore converges to

$$\frac{(1/25)(4/5)^3}{1-4/5} = \frac{64}{625}.$$

(b) Since $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+4}\right) = 1 \ne 0$, the series diverges by the n^{th}-term test.

(c) Since $\displaystyle\frac{2n+1}{n^2+n} = \frac{1}{n} + \frac{1}{n+1}$, terms in the series are larger than terms in the harmonic series, which diverges. It follows that the given series must diverge also.

12 **3.** Find the interval of convergence for the power series

$$\sum_{n=4}^{\infty} \frac{n^a}{n+1}(x+2)^n, \qquad \text{where } a \geq 2 \text{ is an integer.}$$

The radius of convergence of the series is

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{n^a}{n+1}}{\dfrac{(n+1)^a}{n+2}} \right| = \lim_{n\to\infty} \left[\left(\frac{n+2}{n+1}\right) \left(\frac{n}{n+1}\right)^a \right] = 1.$$

The open interval of convergence is therefore $-1 < x+2 < 1$, or, $-3 < x < -1$. At $x = -1$, the series becomes

$$\sum_{n=4}^{\infty} \frac{n^a}{n+1}.$$

Since $\lim_{n\to\infty} \frac{n^a}{n+1} = \infty$, the series diverges by the n^{th}-term test. At $x = -3$, the series becomes

$$\sum_{n=4}^{\infty} \frac{(-1)^n n^a}{n+1}.$$

Since $\lim_{n\to\infty} \frac{(-1)^n n^a}{n+1}$ does not exist, the series diverges by the n^{th}-term test. The interval of convergence is therefore $-3 < x < -1$.

10 **4.** Find the remainder $R_n(2, x)$ when the function $f(x) = e^{4x}$ is expanded with Taylor's remainder formula (about $x = 2$). Verify that $\lim_{n\to\infty} R_n(2, x) = 0$ for all $x \geq 2$.

Since $f^{(n)}(x) = 4^n e^{4x}$, it follows that

$$R_n(2, x) = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-2)^{n+1} = \frac{4^{n+1}e^{4z_n}}{(n+1)!}(x-2)^{n+1}.$$

Since $2 < z_n < x$,

$$|R_n(2, x)| = \frac{4^{n+1}e^{4z_n}}{(n+1)!}|x-2|^{n+1} \leq \frac{4^{n+1}e^{4x}}{(n+1)!}(x-2)^{n+1} = e^{4x}\frac{[4(x-2)]^{n+1}}{(n+1)!}.$$

This approaches zero as $n \to \infty$ for all $x \geq 2$.