MATH2132 Test1 Solutions July 20, 2021

1. Find the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n\, n\, 3^n}{n+1}(x-2)^{2n}.$$

When we set $y = (x-2)^2$, the series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n\, n\, 3^n}{n+1} y^n.$$

The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^n n 3^n}{n+1}}{\dfrac{(-1)^{n+1}(n+1)3^{n+1}}{n+2}} \right| = \frac{1}{3}.$$

The radius of convergence of the original series is therefore $R_x = 1/\sqrt{3}$. The open interval of convergence is

$$-\frac{1}{\sqrt{3}} < x-2 < \frac{1}{\sqrt{3}} \qquad \Longrightarrow \qquad 2-\frac{1}{\sqrt{3}} < x < 2+\frac{1}{\sqrt{3}}.$$

At the end points, the series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n\, n\, 3^n}{n+1}\left(\frac{\pm 1}{\sqrt{3}}\right)^{2n} = \sum_{n=2}^{\infty} \frac{(-1)^n n}{n+1}.$$

Since $\lim_{n\to\infty} \dfrac{(-1)^n n}{n+1}$ does not exist, the series diverges by the n^{th}-term test. The interval of convergence is

$$2-\frac{1}{\sqrt{3}} < x < 2+\frac{1}{\sqrt{3}}.$$

2. Find the Taylor series about $x = 2$ for the function

$$\frac{1}{\sqrt{4+3x}}.$$

Write your answer in sigma notation simplified as much as possible. You must use a method that guarantees that the series converges to the function. What is the open interval of convergence for the series?

x-2

The Taylor series can be obtained with the binomial expansion,

$$\begin{aligned}
\frac{1}{\sqrt{4+3x}} &= \frac{1}{\sqrt{3(x-2)+10}} = \frac{1}{\sqrt{10}}\left[1+\frac{3(x-2)}{10}\right]^{-1/2} \\
&= \frac{1}{\sqrt{10}}\left[1+(-1/2)\frac{3}{10}(x-2)+\frac{(-1/2)(-3/2)}{2!}\frac{3^2}{10^2}(x-2)^2\right. \\
&\qquad \left. +\frac{(-1/2)(-3/2)(-5/2)}{3!}\frac{3^3}{10^3}(x-2)^3+\cdots\right] \\
&= \frac{1}{\sqrt{10}}\left[1-\frac{3}{2\cdot 10}(x-2)+\frac{3^2(3)}{2^2\,10^2\,2!}(x-2)^2-\frac{3^3(1\cdot 3\cdot 5)}{2^3\,10^3\,3!}(x-2)^3+\cdots\right] \\
&= \frac{1}{\sqrt{10}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 3^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n\,10^n\,n!}(x-2)^n\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 3^n (2n)!}{2^{2n}10^{n+1/2}(n!)^2}(x-2)^n.
\end{aligned}$$

The open interval of convergence is

$$-1 < \frac{3(x-2)}{10} < 1 \qquad \Longrightarrow \qquad -\frac{10}{3} < x-2 < \frac{10}{3} \qquad \Longrightarrow \qquad -\frac{4}{3} < x < \frac{16}{3}.$$

3. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)!} x^{2n+2}.$$

If we set $y = x^2$, then

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)!} x^{2n+2} = x^2 \sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)!} y^n.$$

The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^n n 3^n}{(2n)!}}{\dfrac{(-1)^{n+1}(n+1)3^{n+1}}{(2n+2)!}} \right| = \lim_{n\to\infty} \frac{n3^n}{(2n)!} \frac{(2n+2)(2n+1)(2n)!}{(n+1)3^{n+1}} = \infty.$$

Hence, the radius of convergence of the original series is $R_x = \infty$, also. If now set

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)!} x^{2n+2} \quad \text{then} \quad \frac{S(x)}{x^3} = \sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)!} x^{2n-1},$$

provided x is not equal to 0. Integration gives

$$\begin{aligned}
\int \frac{S(x)}{x^3}\, dx &= \sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)(2n)!} x^{2n} + C = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n\, 3^n}{(2n)!} x^{2n} + C \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{3}x)^{2n} + C = \frac{1}{2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{3}x)^{2n} - 1 \right] + C \\
&= \frac{1}{2} [\cos \sqrt{3}x - 1] + C.
\end{aligned}$$

Differentiation gives

$$\frac{S(x)}{x^3} = -\frac{\sqrt{3}}{2} \sin \sqrt{3}x \qquad \Longrightarrow \qquad S(x) = -\frac{\sqrt{3}x^3}{2} \sin \sqrt{3}x.$$

When $x = 0$, the series has sum 0. Since the above formula also gives 0 for $x = 0$, we can write that

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n\, 3^n}{(2n)!} x^{2n+2} = \frac{\sqrt{3}x^3}{2} \sin \sqrt{3}x, \quad -\infty < x < \infty.$$

4. Approximate the value of the integral

$$\int_0^1 \frac{x - \sin x}{x^3} dx$$

accurate to four decimal places. Justify any conclusions that you make.

If we substitute the Maclaurin series for $\sin x$,

$$\begin{aligned}
\int_0^1 \frac{x - \sin x}{x^3} dx &= \int_0^1 \frac{1}{x^3}\left[x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right] dx \\
&= \int_0^1 \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} + \cdots\right) dx \\
&= \left\{\frac{x}{3!} - \frac{x^3}{3 \cdot 5!} - \frac{x^5}{5 \cdot 7!} + \cdots\right\}_0^1 \\
&= \frac{1}{3!} - \frac{1}{3 \cdot 5!} + \frac{1}{5 \cdot 7!} + \cdots
\end{aligned}$$

This alternating series converges since absolute values of terms are decreasing and have limit zero. The sum of the series lies between any two successive partial sums. We calculate that

$$S_1 = 0.166667, \quad S_2 = 0.16389, \quad S_3 = 0.16392.$$

Since S_2 and S_3 agree to 4 decimal places, we can say that to 4 decimal places

$$\int_0^1 \frac{x - \sin x}{x^3} dx \approx 0.1639.$$