MATH2132 Test1 **February 4, 2014** **70 minutes**

1. (a) Determine whether the sequence of functions

$$\{f_n(x)\} = \left\{\frac{3n^2x^2+1}{n^2x^2+2nx+4}\right\}$$

has a limit on the interval $-1 \le x \le 1$. Show your reasoning and all calculations.

(b) Would the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ have a sum? Explain.

(a) $\displaystyle\lim_{n\to\infty}\left\{\frac{3n^2x^2+1}{n^2x^2+2nx+4}\right\} = \lim_{n\to\infty}\frac{3x^2+\frac{1}{n^2}}{x^2+\frac{2x}{n}+\frac{4}{n^2}} = \frac{3x^2}{x^2} = 3$, provided $x \neq 0$.

When $x = 0$, the sequence is $\{f_n(0)\} = \{1/4\}$ with limit $1/4$. Thus,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 3, & x \neq 0 \\ 1/4, & x = 0. \end{cases}$$

(b) Since $\displaystyle\lim_{n\to\infty} f_n(x) \neq 0$ for any x, the series must diverge (by the n^{th}-term test).

10 **2.** Determine whether the following series converge or diverge. If a series converges, find its sum. Justify your conclusions.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+3}}{2^{2n-1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{3n^2+4}\right)$

(a) We write $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+3}}{2^{2n-1}} = \sum_{n=2}^{\infty} \frac{3^3}{2^{-1}}\left(\frac{3}{4}\right)^n$. This shows that the series is geometric with common ration $3/4 < 1$. The series therefore converges, and its sum is

$$\frac{\frac{3^3}{2^{-1}}\left(\frac{3}{4}\right)^2}{1-\frac{3}{4}} = \frac{3^5}{8}(4) = \frac{243}{2}.$$

(b) Since $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{3n^2+4} = \frac{1}{3}$, it follows that $\displaystyle\lim_{n\to\infty} (-1)^n \left(\frac{n^2+1}{3n^2+4}\right)$ does not exist. The series therefore diverges by the n^{th}-term test.

10 **3.** Find the interval of convergence for the power series

$$\sum_{n=3}^{\infty} \frac{(-1)^n n}{a^n} x^{2n}, \qquad \text{where } a > 0 \text{ is a given constant.}$$

If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n n}{a^n} y^n$. Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n n}{a^n}}{\frac{(-1)^{n+1}(n+1)}{a^{n+1}}} \right| = a.$$

Thus, $R_x = \sqrt{a}$, and the open interval of convergence is $-\sqrt{a} < x < \sqrt{a}$. At the end points $x = \pm\sqrt{a}$, the series becomes

$$\sum_{n=3}^{\infty} \frac{(-1)^n n}{a^n}(a^n) = \sum_{n=3}^{\infty} (-1)^n n.$$

Since $\displaystyle\lim_{n\to\infty} (-1)^n n$ does not exist, the series diverges by the n^{th}-term test. The interval of convergence is therefore $-\sqrt{a} < x < \sqrt{a}$.

6 **4.** Find the sum of the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{2^n(2n)!} x^{2n}.$$

Include its interval of convergence.

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^n}{2^n(2n)!} x^{2n} &= \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x}{\sqrt{2}}\right)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x}{\sqrt{2}}\right)^{2n} - 1 + \frac{x^2}{4} \\
&= \cos\left(\frac{x}{\sqrt{2}}\right) - 1 + \frac{x^2}{4}
\end{aligned}$$

The interval of convergence is $-\infty < x < \infty$.

8 **5.** Find the remainder $R_n(1, x)$ when the function $f(x) = \sin 5x$ is expanded with Taylor's remainder formula (about $x = 1$). Verify that $\lim_{n\to\infty} R_n(1, x) = 0$ for all x.

The n^{th} derivative of $\sin 5x$ is 5^n multiplied by either $\pm \sin 5x$ or $\pm \cos 5x$. Hence,

$$R_n(1, x) = \frac{d^{n+1} \sin 5x}{dx^{n+1}}_{|x=z_n} \frac{(x-1)^{n+1}}{(n+1)!} = 5^{n+1} \begin{Bmatrix} \pm \cos 5z_n \\ \pm \sin 5z_n \end{Bmatrix} \frac{(x-1)^{n+1}}{(n+1)!}.$$

When we take absolute values,

$$|R_n(1, x)| = 5^{n+1} \begin{vmatrix} \pm \cos 5z_n \\ \pm \sin 5z_n \end{vmatrix} \frac{|x-1|^{n+1}}{(n+1)!} \le 5^{n+1} \frac{|x-1|^{n+1}}{(n+1)!} = \frac{|5(x-1)|^{n+1}}{(n+1)!}.$$

This approaches 0 as $n \to \infty$ for all x. Thus, $\lim_{n\to\infty} R_n(1, x) = 0$ for all x.