MATH 2132 Test 2 **November 7, 2019** **60 minutes**

Student Name - **Student Number -**

Values

6 **1.** Find a 1-parameter family of solutions for the diferential equation

$$x\frac{dy}{dx} = \cos 2x - 2y.$$

Is your answer a general solution? Explain.

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x}\cos 2x.$$

An integrating factor is $e^{\int (2/x)dx} = e^{2\ln|x|} = x^2$. When we multiply each term in the differential equation by x^2,

$$x^2\frac{dy}{dx} + 2xy = x\cos 2x \qquad \Longrightarrow \qquad \frac{d}{dx}(x^2 y) = x\cos 2x.$$

Integration gives

$$x^2 y = \int x\cos 2x\,dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C.$$

Hence,

$$y(x) = \frac{1}{2x}\sin 2x + \frac{1}{4x^2}\cos 2x + \frac{C}{x^2}.$$

Since the differential equation is linear, this one-parameter family of solutions is a general solution.

10 **2.** (a) Find a 1-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = \frac{3x^2}{1-y}.$$

(b) Find the solution of the differential equation that also satisfies the condition $y(0) = 3$. Determine the largest possible interval on which your solution satisfies the differential equation.

(a) The equation is separable,

$$(1-y)\,dy = 3x^2\,dx.$$

and a 1-parameter family of solutions is defined implicitly by

$$y - \frac{y^2}{2} = x^3 + C.$$

(b) The condition $y(0) = 3$ requires $3 - 9/2 = C$. Thus,

$$\begin{aligned} y - \frac{y^2}{2} &= x^3 - \frac{3}{2} \\ y^2 - 2y + (2x^2 - 3) &= 0 \\ y &= \frac{2 \pm \sqrt{4 - 4(2x^3 - 3)}}{2} = 1 \pm \sqrt{4 - 2x^3}. \end{aligned}$$

Only the positive root satisfies the initial condition. The solution is therefore

$$y(x) = 1 + \sqrt{4 - 2x^3}.$$

It is defined for $4 - 2x^3 \geq 0 \implies x \leq 2^{1/3}$. But, it satisfies the differential equation only for $x < 2^{1/3}$.

13 **3.** Find a general solution for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 3y = 2x + \sin 2x.$$

The auxiliary equation $m^2 + 3m + 3 = 0$ has roots $m = \dfrac{-3 \pm \sqrt{9-12}}{2} = -\dfrac{3}{2} \pm \dfrac{\sqrt{3}i}{2}$. A general solution of the associated homogeneous equation is

$$y_h(x) = e^{-3x/2}\left(C_1 \cos\frac{\sqrt{3}x}{2} + C_2 \sin\frac{\sqrt{3}x}{2}\right).$$

If we substitute a particular soluton $y_p(x) = Ax + B + C\sin 2x + D\cos 2x$ into the nonhomogeneous equation

$$\begin{aligned}(-4C\sin 2x - 4D\cos 2x) + 3(A + 2C\cos 2x - 2D\sin 2x)\\ + 3(Ax + B + C\sin 2x + D\cos 2x) = 2x + \sin 2x.\end{aligned}$$

When we equate coefficients:

x: $3A = 2,$
1: $3A + 3B = 0,$
$\cos 2x$: $-4D + 6C + 3D = 0,$
$\sin 2x$: $-4C - 6D + 3C = 1.$
These imply that $A = 2/3$, $B = -2/3$, $C = -1/37$, and $D = -6/37$. Thus,

$$y_p(x) = \frac{2x}{3} - \frac{2}{3} - \frac{1}{37}\sin 2x - \frac{6}{37}\cos 2x.$$

A general solution is

$$y(x) = e^{-3x/2}\left(C_1 \cos\frac{\sqrt{3}x}{2} + C_2 \sin\frac{\sqrt{3}x}{2}\right) + \frac{2x}{3} - \frac{2}{3} - \frac{1}{37}\sin 2x - \frac{6}{37}\cos 2x.$$

6 **4.** If the roots of the auxiliary equation for the differential equation

$$\phi(D)y = 2x^3 - 1 + e^x \sin x + e^{4x}$$

are $m = 0,\ 0,\ \pm 3,\ 2 \pm i,\ 2 \pm i,\ 4,\ 4,\ 4$, what is the form of a particular solution as predicted by the method of undetermined coefficients?

$$y_h(x) = C_1 + C_2 x + C_3 e^{3x} + C_4 e^{-3x} + e^{2x}[(C_5 + C_6 x)\cos x + (C_7 + C_8 x)\sin x] + (C_9 + C_{10} x + C_{11} x^2)e^{4x}$$

$$y_p(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ee^x \sin x + Fe^x \cos x + Gx^3 e^{4x}$$

6 **5.** Two substances A and B react to form a third substance C in such a way that 3 grams of A react with 4 grams of B to produce 7 grams of C. The rate at which C is formed is proportional to the product of the amounts of A and B present in the mixture. Originally a container contains 10 grams of A and 20 grams of C. If 30 grams of B are added to the container, set up, but do **NOT** solve, an initial-value problem for the amount of C in the container after B is added.

Let $x(t)$ represent the number of grams of C in the mixture where t is time taking $t = 0$ when B is added to the container. The initial-value problem for $x(t)$ is

$$\frac{dx}{dt} = k\left[10 - \frac{3}{7}(x - 20)\right]\left[30 - \frac{4}{7}(x - 20)\right], \qquad x(0) = 20.$$