MATH2132 Test 2 **November 17, 2022** **60 minutes**

Values

10 **1.** Find, in explicit form, the solution of the initial-value problem

$$\frac{dy}{dx} = \frac{1+y^3}{xy^2+y^2}, \qquad y(0) = 1.$$

The dfferential equation can be separated

$$\frac{y^2\,dy}{1+y^3} = \frac{dx}{x+1}, \qquad (y \neq -1).$$

A one-parameter of solutions is defined implicitly by

$$\frac{1}{3}\ln|1+y^3| = \ln|x+1| + C.$$

For an explicit solution, we write

$$|1+y^3| = e^{3C}|x+1|^3 \qquad \Longrightarrow \qquad 1+y^3 = D|x+1|^3, \qquad D = \pm e^{3C}.$$

The initial condition requires $2 = D$, and therefore

$$1+y^3 = 2|x+1|^3 \qquad \Longrightarrow \qquad y = (2|x+1|^3-1)^{1/3}.$$

7 **2.** A tank originally contains 1000 litres of water in which 10 kilograms of salt has been dissolved. A mixture containing 2 kilograms of salt for each 100 litres of water is added to the tank at 10 millilitres per second. At the same time 15 millilitres of well-stirred mixture is removed from the tank. Set up, but do **NOT** solve, an initial-value problem for the number of grams of salt in the tank as a function of time. For how long is the differential equation valid?

If we let $S(t)$ be the number of grams of salt in the tank as a function of time t in seconds, then

$$\frac{dS}{dt} = (\text{rate salt enters}) - (\text{rate salt leaves}) = \frac{1}{5} - \frac{15S}{10^6-5t}, \qquad S(0) = 10\,000.$$

The differential equation is valid as long as there is water in the tank. This ends when $10^6 - 5t = 0$, and this gives $t = 200,000$ seconds.

12 **3.** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n)!} x^{2n}.$$

Justify all steps in your solution.

If we set $y = x^2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n)!} y^n$. The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n n}{(2n)!}}{\frac{(-1)^{n+1}(n+1)}{(2n+2)!}} \right| = \lim_{n\to\infty} \left[\frac{n(2n+2)(2n+1)(2n)!}{(2n)!(n+1)} \right] = \infty.$$

Thus, $R_x = \infty$ also. If we set $S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n)!} x^{2n}$, then

$$\frac{S(x)}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n)!} x^{2n-1}, \qquad x \neq 0.$$

Because the radius of convergence is infinite, we can integrate this series term-by-term,

$$\int \frac{S(x)}{x}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n)!(2n)} x^{2n} + C = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + C = \frac{1}{2}(\cos x - 1) + C.$$

Differentiation gives

$$\frac{S(x)}{x} = -\frac{1}{2} \sin x \qquad \Longrightarrow \qquad S(x) = -\frac{x}{2} \sin x, \qquad x \neq 0.$$

When $x = 0$, $S(0) = 0$, and the sum of the series at $x = 0$ is also 0. Hence, we can write that

$$S(x) = -\frac{x}{2} \sin x, \qquad -\infty < x < \infty.$$

11 **4.** (a) Find, in sigma notation, a series representing

$$\int_0^x e^{-t^2}\,dt.$$

(b) Suppose your series in part (a) is truncated when $n = N$. Find an inequality that N must satisfy if the maximum error on the interval $0 \le x \le 1$ must be less than 10^{-8}. Do **NOT** solve the inequality. Justify all statements.

(a) Using the Maclaurin series for e^{-t^2},

$$\int_0^x e^{-t^2}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}\right) dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}\right) dt$$
$$= \left\{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} t^{2n+1}\right\}_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}.$$

(b) The series is alternating when $0 \le x \le 1$. Since the sequence $\left\{\dfrac{x^{2n+1}}{n!(2n+1)}\right\}$ is decreasing and has limit 0 for any x in the interval $0 \le x \le 1$, the series converges by the alternating series test. If the series is truncated when $n = N$, the absolute value of the maximum error is

$$\frac{1}{(N+1)!(2N+3)} x^{2N+3} \le \frac{1}{(N+1)!(2N+3)}.$$

For this to be less than 10^{-8}, we require

$$\frac{1}{(N+1)!(2N+3)} < 10^{-8}.$$