Solutions to MATH2132 Test2 Fall 2023

Values

1. (a) Find an explicitly-defined, 1-parameter family of solutions for the differential equation

$$(y\tan x - 1 - x\tan x + \sec x)dx + dy = 0.$$

(b) Is your family a general solution? Explain.

(a) $\dfrac{dy}{dx} + y\tan x = 1 + x\tan x - \sec x$

Since this is a linear first-order differential equation, an integrating factor is

$$e^{\int \tan x\, dx} = e^{-\ln|\cos x|} = |\sec x|.$$

Whether $\sec x > 0$ or $\sec x < 0$, we get

$$\sec x\frac{dy}{dx} + y\sec x\tan x = \sec x(1 + x\tan x - \sec x),$$

or,

$$\frac{d}{dx}(y\sec x) = \sec x + x\sec x\tan x - \sec^2 x.$$

Integration gives

$$y\sec x = x\sec x - \tan x + C.$$

An explicit solution is

$$y(x) = x - \frac{\tan x}{\sec x} + C\cos x = x - \sin x + C\cos x.$$

(b) Since the differential equation is linear, and we have a 1-parameter family of solutions, the family must be a general solution.

2. What is the form of a particular solution of the differential equation

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 20y = x^2e^{2x} + x\cos x - x^3$$

as predicted by the method of undetermined coefficients? Do **NOT** evaluate the coefficients.

The auxiliary equation is

$$0 = m^3 + m^2 - 16m + 20 = (m-2)(m^2+3m-10) = (m-2)^2(m+5).$$

Solutions are $m = 2, 2, -5$, from which a general solution of the associated homogeneous equation is

$$y_h(x) = (C_1 + C_2x)e^{2x} + C_3e^{-5x}.$$

A particular solution is of the form

$$y_p(x) = Ax^4e^{2x} + Bx^3e^{2x} + Cx^2e^{2x} + Dx\cos x + Ex\sin x + F\cos x + G\sin x + Hx^3 + Ix^2 + Jx + K.$$

3. Evaluate the sum

$$\sum_{n=1}^{\infty} n^2 x^{n-1}.$$

The radius of convergence of the series is $R = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right| = 1$. If we set

$$S(x) = \sum_{n=1}^{\infty} n^2 x^{n-1},$$

and integrate term-by-term,

$$\int S(x)\,dx = \sum_{n=1}^{\infty} n x^n + C.$$

Division by x gives

$$\frac{1}{x}\int S(x)\,dx = \sum_{n=1}^{\infty} n x^{n-1} + \frac{C}{x}, \qquad \text{provided } x \neq 0.$$

Integration now gives

$$\int \left[\frac{1}{x}\int S(x)\,dx\right] dx = \sum_{n=1}^{\infty} x^n + C\ln|x| + D = \frac{x}{1-x} + C\ln|x| + D.$$

We now differentiate with respect to x,

$$\frac{1}{x}\int S(x)\,dx = \frac{(1-x)(1)-x(-1)}{(1-x)^2} + \frac{C}{x} = \frac{1}{(1-x)^2} + \frac{C}{x}.$$

Multiplication be x gives

$$\int S(x)\,dx = \frac{x}{(1-x)^2} + C.$$

Differentiation now gives

$$S(x) = \frac{(1-x)^2(1) - x(2)(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3}.$$

Since the sum of the series at $x = 0$ is 1, and this is $S(0)$, the sum of the series is

$$S(x) = \frac{1+x}{(1-x)^3}, \quad -1 < x < 1.$$

4. The first three terms in the Maclaurin series for the function $1/(1-x)^3$ are

$$1 + 3x + 6x^2.$$

Find an expression representing the maximum possible error when this approximaton is used on the interval $0 \le x \le 0.2$.

The absolute value of the error is

$$|R_2(0,x)| = \left| \frac{d^3}{dx^3} \left[\frac{1}{(1-x)^3} \right]_{x=z} \right| \frac{|x|^3}{3!} = \left| \frac{60}{(1-z)^6} \right| \frac{x^3}{6} \le \frac{10}{|1-x|^6} x^3 \le \frac{10}{(1-0.2)^6} (0.2)^3.$$