MATH2132 Test 2 **June, 2014** **60 minutes**

Student Name - **Student Number -**

Values

14 **1.** Find the Taylor series about $x = -2$ for the function $(x+2)^2 \ln(x+5)$. Use a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible. What is the open of convergence of the series written in the form $a < x < b$?

$\boxed{\text{x+2}}$

$$\frac{1}{x+5} = \frac{1}{(x+2)+3} = \frac{1/3}{1+\frac{x+2}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x+2}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x+2)^n,$$

valid for

$$\left|-\frac{x+2}{3}\right| < 1 \implies |x+2| < 3 \implies -3 < x+2 < 3 \implies -5 < x < 1.$$

Integration with respect to x gives

$$\ln|x+5| = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}(n+1)}(x+2)^{n+1} + C.$$

When we set $x = -2$, we get $\ln 3 = C$. Hence,

$$\ln(x+5) = \ln 3 + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}(n+1)}(x+2)^{n+1}.$$

Multiplication by $(x+2)^2$ now gives

$$\begin{aligned}(x+2)^2\ln(x+5) &= (\ln 3)(x+2)^2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}(n+1)}(x+2)^{n+3}\\ &= (\ln 3)(x+2)^2 + \sum_{n=3}^{\infty}\frac{(-1)^{n-3}}{3^{n-2}(n-2)}(x+2)^n.\end{aligned}$$

The open interval of convergence is $-5 < x < 1$.

8 **2.** Evaluate

$$\lim_{x\to 3} \frac{(x-3) - \dfrac{(x-3)^3}{6} - \sin(x-3)}{(x-3)^5}.$$

If we denote the limit by L, then

$$\begin{aligned} L &= \lim_{x\to 3} \frac{(x-3) - \dfrac{(x-3)^3}{6} - \left[(x-3) - \dfrac{(x-3)^3}{3!} + \dfrac{(x-3)^5}{5!} - \dfrac{(x-3)^7}{7!} + \cdots\right]}{(x-3)^5} \\ &= \lim_{x\to 3} \left[-\frac{1}{5!} + \frac{(x-3)^2}{7!} + \cdots\right] \\ &= -\frac{1}{5!}. \end{aligned}$$

9 **3.** Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{2n}.$$

If we set $y = x^2$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} y^n$. The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n 2^n}{n+1}}{\frac{(-1)^{n+1}2^{n+1}}{n+2}} \right| = \frac{1}{2}.$$

Consequently, $R_x = 1/\sqrt{2}$. If we set $S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{2n}$, then

$$x^2 S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{2n+2}.$$

Differentiation with respect to x gives

$$\frac{d}{dx}\left[x^2 S(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (2n+2)}{n+1} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} x^{2n+1} = \sum_{n=0}^{\infty} 2x(-2x^2)^n.$$

Since this is a geometric series,

$$\frac{d}{dx}\left[x^2 S(x)\right] = \frac{2x}{1-(-2x^2)} = \frac{2x}{1+2x^2},$$

valid for $|-2x^2| < 1 \quad \Longrightarrow \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Integration with respect to x gives

$$x^2 S(x) = \frac{1}{2}\ln|1+2x^2| + C.$$

If we set $x = 0$, we obtain $0 = C$. Hence,

$$x^2 S(x) = \frac{1}{2}\ln|1+2x^2|.$$

When $x \neq 0$, we obtain

$$S(x) = \frac{1}{2x^2}\ln(1+2x^2).$$

When $x = 0$, the sum of the series is $S(0) = 1$. Thus,

$$S(x) = \begin{cases} \frac{1}{2x^2}\ln(1+2x^2), & -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \\ 1, & x = 0. \end{cases}$$

9 **4.** Find in explicit form the solution of the initial value problem

$$y\frac{dy}{dx} = \frac{x}{(y^2+1)^3}, \qquad y(0) = 1.$$

What is the domain of the solution?

The equation can be separated,

$$y(y^2+1)^3\,dy = x\,dx.$$

A 1-parameter family of solutions is defined implicitly by

$$\begin{aligned} \int y(y^2+1)^3\,dy &= \int x\,dx \\ \frac{1}{8}(y^2+1)^4 &= \frac{x^2}{2} + C. \end{aligned}$$

The initial condition requires $\frac{1}{8}(2)^4 = C$ so that $C = 2$. The solution is defined implicitly by

$$\begin{aligned} \frac{1}{8}(y^2+1)^4 &= \frac{x^2}{2} + 2 \\ (y^2+1)^4 &= 4x^2 + 16 \\ y^2 + 1 &= \pm(4x^2+16)^{1/4}. \end{aligned}$$

Since the left side is positive, we must choose the positive sign,

$$\begin{aligned} y^2 + 1 &= (4x^2+16)^{1/4} \\ y^2 &= (4x^2+16)^{1/4} - 1 \\ y &= \pm\sqrt{(4x^2+16)^{1/4} - 1}. \end{aligned}$$

Only with the positive sign is the initial condition satisfied. Thus,

$$y(x) = \sqrt{(4x^2+16)^{1/4} - 1}.$$

Since $(4x^2+16)^{1/4} \geq 2$ for all x, the solution is valid for all x.