MATH 2132 Test 2 **March 4, 2014** **75 minutes**

Student Name - **Student Number -**

Values

12 **1.** (a) Find the Taylor series about $x = 3$ for the function $\sqrt{2+x}$. Express your answer in sigma notation simplified as much as possible. You must use a technique that ensures that the series converges to the function.

(b) What is the radius of convergence of the series?

$\boxed{\text{x-3}}$

$$\begin{aligned}
\sqrt{2+x} = \sqrt{5+(x-3)} &= \sqrt{5}\left[1+\frac{x-3}{5}\right]^{1/2} \\
&= \sqrt{5}\left[1+(1/2)\left(\frac{x-3}{5}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{x-3}{5}\right)^2+\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x-3}{5}\right)^3+\cdots\right] \\
&= \sqrt{5}\left[1+\frac{1}{2\cdot 5}(x-3)-\frac{1}{2^2 2! 5^2}(x-3)^2+\frac{1\cdot 3}{2^3 3! 5^3}(x-3)^3-\frac{1\cdot 3\cdot 5}{2^4 4! 5^4}(x-3)^4+\cdots\right] \\
&= \sqrt{5}\left[1+\frac{1}{10}(x-3)+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{2^n n! 5^n}(x-3)^n\right] \\
&= \sqrt{5}\left[1+\frac{1}{10}(x-3)+\sum_{n=2}^{\infty}\frac{[1\cdot 2\cdot 3\cdots(2n-2)]}{2^n n! 5^n[2\cdot 4\cdot 6\cdots(2n-2)]}(x-3)^n\right] \\
&= \sqrt{5}+\sum_{n=1}^{\infty}\frac{\sqrt{5}(-1)^{n+1}(2n-2)!}{2^{2n-1}5^n n!(n-1)!}(x-3)^n.
\end{aligned}$$

(b) Since the open interval of convergence is defined by $\left|\frac{x-3}{5}\right| < 1 \quad \Longrightarrow \quad |x-3| < 5$, the radius of convergence is 5.

10 **2.** (a) Find a series of constants that converges to the value of the integral

$$\int_0^{1/2} \frac{x - \sin x}{x^3} dx.$$

(b) Explain how you would use the series to obtain an approximation to the integral to 6 decimal places. Justify all statements.

(a)

$$\begin{aligned}
\int_0^{1/2} \frac{x - \sin x}{x^3} dx &= \int_0^{1/2} \frac{1}{x^3} \left[x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) \right] dx \\
&= \int_0^{1/2} \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right) dx \\
&= \left\{ \frac{x}{3!} - \frac{x^3}{3 \cdot 5!} + \frac{x^5}{5 \cdot 7!} - \cdots \right\}_0^{1/2} \\
&= \frac{1}{2 \cdot 3!} - \frac{1}{3 \cdot 2^3 \cdot 5!} + \frac{1}{5 \cdot 2^5 \cdot 7!} - \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)! 2^{2n-1}}
\end{aligned}$$

(b) Since the series is alternating, and absolute values of terms are decreasing and approaching zero, the series converges by the alternating series test. Since the sum of the series lies between any two successive partial sums, we should calculate partial sums until two successive ones agree to 6 decimal places.

8 **3.** Evaluate

$$\sum_{n=2}^{\infty} 2^n (n-1) x^n.$$

Justify all steps in your solution.

The radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{2^n(n-1)}{2^{n+1}n} \right| = \dfrac{1}{2}$.

If we let $S(x) = \sum_{n=2}^{\infty} 2^n (n-1) x^n$, then

$$\frac{1}{x^2} S(x) = \sum_{n=2}^{\infty} 2^n (n-1) x^{n-2}, \qquad \text{provided } x \neq 0.$$

Integration gives

$$\int \frac{1}{x^2} S(x)\, dx = \sum_{n=2}^{\infty} 2^n x^{n-1} + C = \frac{4x}{1-2x} + C.$$

Differentiation now gives

$$\frac{1}{x^2} S(x) = \frac{(1-2x)(4) - 4x(-2)}{(1-2x)^2} = \frac{4}{(1-2x)^2}.$$

Thus,

$$S(x) = \frac{4x^2}{(1-2x)^2}, \qquad x \neq 0.$$

When $x = 0$, the sum of the series is 0. Since $S(0) = 0$ also, we can write that

$$\sum_{n=2}^{\infty} 2^n (n-1) x^n = \frac{4x^2}{(1-2x)^2}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

10 **4.** Find, in explicit form $y = f(x)$, a one-parameter family of solutions to the differential equation

$$y^{-1}\frac{dy}{dx} - y\cos x = -\cos x.$$

Does your family of solutions have any singular solutions? Can it (or they) be added to the family?

The equation is separable

$$\frac{1}{y}\frac{dy}{dx} = y\cos x - \cos x = \cos x(y-1) \quad \Longrightarrow \quad \frac{dy}{y(y-1)} = \cos x\, dx, \qquad y \neq 1.$$

A one-parameter family of solutions is defined implicitly by

$$\int \frac{1}{y(y-1)}dy = \int \cos x\, dx + C.$$

Thus,

$$\sin x + C = \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \ln|y-1| - \ln|y| = \ln\left|\frac{y-1}{y}\right|.$$

If we exponentiate,

$$\begin{aligned}
\left|\frac{y-1}{y}\right| &= e^{\sin x + C} \\
\frac{y-1}{y} &= \pm e^C e^{\sin x} = De^{\sin x}, \qquad D = \pm e^C \\
y - 1 &= Dye^{\sin x} \\
y(De^{\sin x} - 1) &= -1 \\
y &= \frac{1}{1 - De^{\sin x}}.
\end{aligned}$$

$y = 0$ is not a solution of the original differential equation. $y = 1$ is a solution. Since $D = \pm e^C \neq 0$, $y = 1$ is not in the family, and is therefore a singular solution of the family. However, if we allow D to be zero, then $y = 1$ is contained in the family, and is no longer singular.