MATH 2132 Test 2 **March 21, 2017** **60 minutes**

Student Name - **Student Number -**

Values

8 **1.** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{(2n+1)!} x^{2n+1}.$$

If we set $y = x^2$, then

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{(2n+1)!} x^{2n+1} = x \sum_{n=1}^{\infty} \frac{n(-1)^n}{(2n+1)!} y^n.$$

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{n(-1)^n}{(2n+1)!}}{\frac{(n+1)(-1)^{n+1}}{(2n+3)!}} \right| = \lim_{n\to\infty} \frac{n(2n+3)(2n+2)(2n+1)!}{(n+1)(2n+1)!} = \infty.$$

Thus, $R_x = \infty$. If we set $S(x) = \sum_{n=1}^{\infty} \frac{n(-1)^n}{(2n+1)!} x^{2n+1}$, then

$$\frac{1}{x^2} S(x) = \sum_{n=1}^{\infty} \frac{n(-1)^n}{(2n+1)!} x^{2n-1}, \qquad x \neq 0$$

$$\int \frac{1}{x^2} S(x)\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{2(2n+1)!} x^{2n} + C$$

$$x \int \frac{1}{x^2} S(x)\, dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + Cx = \frac{1}{2}(\sin x - x) + Cx$$

$$\int \frac{1}{x^2} S(x)\, dx = \frac{1}{2x} \sin x - \frac{1}{2} + C$$

$$\frac{1}{x^2} S(x) = -\frac{1}{2x^2} \sin x + \frac{1}{2x} \cos x$$

$$S(x) = -\frac{1}{2} \sin x + \frac{x}{2} \cos x.$$

When $x = 0$, the sum of the series is 0. Since the above formula also gives $S(0) = 0$, we can write that

$$S(x) = \frac{x}{2} \cos x - \frac{1}{2} \sin x, \quad -\infty < x < \infty.$$

6 **2.** Determine whether the following series converges or diverges. Justify all statements.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{2n^2+3} \right)$$

The series is alternating. If we set $f(x) = \dfrac{x+1}{2x^2+3}$, then

$$f'(x) = \frac{(2x^2+3)(1) - (x+1)(4x)}{(2x^2+3)^2} = \frac{3-4x-2x^2}{(2x^2+3)^2}.$$

Since this is negative for $x \geq 1$, it follows that the sequence $\left\{ \dfrac{n+1}{2n^2+3} \right\}$ is decreasing. It is also clear that

$$\lim_{n\to\infty} \frac{n+1}{2n^2+3} = 0.$$

Thus, the series converges by the alternating series test.

12 **3.** (a) Find an infinite series of constants that converges to the value of the definite integral

$$\int_0^{1/2} \sqrt{1+x^6}\, dx.$$

Express the series in sigma notation, but do **NOT** simplify the result.

(b) Explain how you would use the series to find an 8 decimal approximation to the definite integral.

(a)

$$\int_0^{1/2} \sqrt{1+x^6}\, dx = \int_0^{1/2} \left[1 + (1/2)x^6 + \frac{(1/2)(-1/2)}{2!}(x^6)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}(x^6)^2 + \cdots \right] dx$$

$$= \left\{ x + \frac{x^7}{2\cdot 7} - \frac{x^{13}}{2^2\, 2!\, 13} + \frac{3x^{19}}{2^3\, 3!\, 19} - \cdots \right\}_0^{1/2}$$

$$= \frac{1}{2} + \frac{1}{2\, 2^7\, 7} - \frac{1}{2^2\, 2^{13}\, 2!\, 13} + \frac{3}{2^3\, 2^{19}\, 3!\, 19} - \frac{1\cdot 3\cdot 5}{2^4\, 2^{25}\, 4!\, 25} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2^8\, 7} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}[1\cdot 3\cdot 5 \cdots (2n-3)]}{2^{7n+1} n!(6n+1)}.$$

(b) The series is alternating after the second term. Since absolute values of terms decrease and approach zero, the series converges by the alternating series test. Calculate partial sums of the series until two consecutive partial sums agree to 8 decimal places.

8 **4.** Find an explicit solution for the initial-value problem

$$y\frac{dy}{dx} = 2x\sqrt{4+y^2}, \quad y(0) = 1.$$

The equation is separable

$$\frac{y}{\sqrt{4+y^2}}dy = 2x\,dx.$$

A one-parameter family of solutions is defined implicilty by

$$\begin{aligned}\int \frac{y}{\sqrt{4+y^2}}dx &= \int 2x\,dx + C\\ \sqrt{4+y^2} &= x^2 + C\end{aligned}$$

Since $y(0) = 1$, we get $\sqrt{5} = C$. Thus,

$$\begin{aligned}\sqrt{4+y^2} &= x^2 + \sqrt{5}\\ 4 + y^2 &= (x^2+\sqrt{5})^2\\ y^2 &= (x^2+\sqrt{5})^2 - 4\\ y &= \pm\sqrt{(x^2+\sqrt{5})^2 - 4}.\end{aligned}$$

Since $y(0) = 1$ is not satisfied when the negative root is chosen, $y = \sqrt{(x^2+\sqrt{5})^2 - 4}$.

7 **5.** A spherical candy has radius 1 centimetre. When it is submerged in water, it dissolves, but always remains spherical. If it dissolves at a rate proportional to its surface area, find its radius as a function of time. You might find useful the following formulas for the volume and surface area of a sphere:

$$V = \frac{4}{3}\pi r^3, \qquad A = 4\pi r^2$$

Since dV/dt represents the rate at which the candy dissolves,

$$\begin{aligned}\frac{dV}{dt} &= kA, \quad \text{where } k \text{ is a constant}\\ \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) &= k(4\pi r^2)\\ 4\pi r^2\frac{dr}{dt} &= 4\pi r^2 k\\ \frac{dr}{dt} &= k\\ r(t) &= kt + C\end{aligned}$$

If we choose $t = 0$ when $r = 1$, then $1 = C$, and $r(t) = kt + 1$.