MATH2132 Test 3 **November 24, 2011** **60 minutes**

Student Name - **Student Number -**

Values

14 **1.** (a) Find a 2-parameter family of solutions of the differential equation

$$xy'' - 3y' = x^5.$$

(b) Can there be any singular solutions to your family of solutions in part (a)? Explain.

(a) Since y is explicitly missing from the differential equation, we set $v = y'$ and $v' = y''$,

$$x\frac{dv}{dx} - 3v = x^5 \qquad \Longrightarrow \qquad \frac{dv}{dx} - \frac{3}{x}v = x^4.$$

An integrating factor for this linear first-order equation is

$$e^{\int (-3/x)\,dx} = e^{-3\ln|x|} = \frac{1}{|x|^3}.$$

When $x > 0$ we multiply by $1/x^3$, and when $x < 0$, we multiply by $-1/x^3$. In both cases, we get

$$\frac{1}{x^3}\frac{dv}{dx} - \frac{3}{x^4}v = x$$

$$\frac{d}{dx}\left(\frac{v}{x^3}\right) = x$$

$$\frac{v}{x^3} = \frac{x^2}{2} + C$$

$$v = \frac{dy}{dx} = \frac{x^5}{2} + Cx^3$$

$$y = \frac{x^6}{12} + \frac{Cx^4}{4} + D$$

(b) Since the given equation is linear, a 2-parameter family of solutions is a general solution, and there can be no singular solutions.

14 **2.** Two substances A and B react to form a third substance C in such a way that 1 gram of A reacts with 1 gram of B to produce 2 grams of C. The rate at which C is formed is proportional to the product of the amounts of A and B present in the mixture. If 10 grams of A and 10 grams of B are originally brought together at time $t = 0$, find the amount of C present in the mixture as a function of time.

If we let $C(t)$ be the number of grams of C in the mixture at any given time, then

$$\frac{dC}{dt} = k\left(10 - \frac{C}{2}\right)\left(10 - \frac{C}{2}\right) = \frac{k}{4}(20 - C)^2, \quad C(0) = 0.$$

This equation is separable

$$\frac{dC}{(20 - C)^2} = \frac{k}{4}dt.$$

A 1-parameter family of solutions is defined implicitly by

$$\frac{1}{20 - C} = \frac{kt}{4} + D.$$

Since $C(0) = 0$,

$$\frac{1}{20} = D.$$

Thus,

$$\frac{1}{20 - C} = \frac{kt}{4} + \frac{1}{20} \qquad \Longrightarrow \qquad C(t) = \frac{100kt}{1 + 5kt} \text{ grams.}$$

7 **3.** Let $\phi(m) = 0$ be the auxiliary equation associated with the differential equation $\phi(D)y = 0$. It is known that

$$\phi(m) = (m + 1)(m - 7)^3(m^2 - 4m + 13)^2.$$

What is a general solution of the differential equation?

Since $m^2 - 4m + 13 = 0$ when $m = \dfrac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$, solutions of the auxiliary equation are

$$m = -1, 7, 7, 7, 2 \pm 3i, 2 \pm 3i.$$

A general solution of the differential equation is

$$y(x) = C_1e^{-x} + (C_2 + C_3x + C_4x^2)e^{7x} + e^{2x}[(C_5 + C_6x)\cos 3x + (C_7 + C_8x)\sin 3x].$$

15 **4.** Find a general solution of the differential equation

$$y'' - 4y' - 5y = 8xe^x.$$

The auxiliary equation is

$$0 = m^2 - 4m - 5 = (m + 1)(m - 5),$$

with solutions $m = -1, 5$. A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 e^{-x} + C_2 e^{5x}.$$

If we substitute a particular solution $y_p(x) = Axe^x + Be^x$ into the differential equation, we get

$$[2Ae^x + Axe^x + Be^x] - 4[Axe^x + Ae^x + Be^x] - 5[Axe^x + Be^x] = 8xe^x.$$

We equate coefficients of xe^x and e^x:

$$\begin{aligned} A - 4A - 5A &= 8, \\ 2A + B - 4A - 4B - 5B &= 0. \end{aligned}$$

Solutions are $A = -1$ and $B = 1/4$. A particular solution is $y_p(x) = -xe^x + e^x/4$. A general solution of the given equation is therefore

$$y(x) = C_1 e^{-x} + C_2 e^{5x} - xe^x + \frac{1}{4}e^x.$$