MATH2132 Test 3 **November 22, 2018** **60 minutes**

Student Name - **Student Number -**

Values

6 **1.** Determine which of the following differential equations are linear and which are nonlinear. If a differential equation is nonlinear, explain why it is. If a differential equation is linear, is it homogeneous or nonhomogeneous?

(a) $\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} + 3xy = x^2 \sin 2x$

(b) $\dfrac{d^3y}{dx^3} + y \sin x = e^{3x}$

(c) $\dfrac{d^{10}y}{dx^{10}} + x^5 \dfrac{d^4y}{dx^4} + 5y = 0$

(a) This equation is nonlinear because of the term $y\dfrac{dy}{dx}$.

(b) This equation is linear and nonhomogeneous.

(c) This equation is linear and homogeneous.

8 **2.** If the roots of the auxiliary equation associated with the differential equation

$$\phi(D)y = 2x^2e^x - e^{3x}\cos 2x + x - e^{2x}$$

are $m = 1, 1, 1, 3 \pm 2i, 2 \pm i, 2 \pm i, 0, 0$, what is the form of a particular solution of the differential equation as predicted by the method of undetermined coefficients?

Since

$$y_h(x) = (C_1 + C_2x + C_3x^2)e^x + e^{3x}(C_4 \cos 2x + C_5 \sin 2x) + e^{2x}[(C_6 + C_7x)\cos x + (C_8 + C_9x)\sin x] + C_{10} + C_{11}x,$$

$$y_p(x) = Ax^5e^x + Bx^4e^x + Cx^3e^x + e^{3x}(Dx\cos 2x + Ex\sin 2x) + Fx^3 + Gx^2 + He^{2x}.$$

15 **3.** Find a 3-parameter family of solutions for the differential equation

$$y''' + y'' + 9y' - 5y = e^{2x} - 3x.$$

Is your solution a general solution? Explain.

The auxiliary equation is

$$0 = 2m^3 + m^2 + 9m - 5 = (2m - 1)(m^2 + m + 5),$$

with roots $m = 1/2$ and $m = \dfrac{-1 \pm \sqrt{1 - 20}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{19}i}{2}$. Thus,

$$y_h(x) = C_1 e^{x/2} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{19}x}{2} + C_3 \sin\frac{\sqrt{19}x}{2}\right).$$

If we assume a particular solution of the form $y_p(x) = Ae^{2x} + Bx + C$, and substitute into the differential equation,

$$2(8Ae^{2x}) + (4Ae^{2x}) + 9(2Ae^{2x} + 8) - 5(Ae^{2x} + Bx + C) = e^{2x} - 3x.$$

When we equation coefficients:

$e^{2x}:\quad 16A + 4A + 18A - 5A = 1 \quad\Longrightarrow\quad A = 1/33$

$x:\quad -5B = -3 \quad\Longrightarrow\quad B = 3/5$

$1:\quad 9B - 5C = 0 \quad\Longrightarrow\quad C = 27/25$

Thus, $y_p(x) = \dfrac{1}{33}e^{2x} + \dfrac{3x}{5} + \dfrac{27}{25}$, and a 3-parameter family of solutions is

$$y(x) = y_h(x) + y_p(x) = C_1 e^{x/2} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{19}x}{2} + C_3 \sin\frac{\sqrt{19}x}{2}\right) + \frac{1}{33}e^{2x} + \frac{3x}{5} + \frac{27}{25}.$$

Since the differential equation is linear and third-order, and we have a 3-parameter family of solutions, the solution is general.

11 **4.** A 2 kilogram mass hangs motionless on the end of a spring with constant 18 newtons per metre. It is set into vertical motion by giving it a downward velocity of 2 metres per second. During its subsequent motion, it experiences damping in magnitude equal to 20 times its speed.

(a) Find the displacement of the mass as a function of time.

(b) Is the motion underdamped, critically damped, or overdamped?

(c) Find the maximum distance that the mass achieves from its equilibrium position.

(a) The initial-value problem for displacements $x(t)$ from equilibrium is

$$2\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 18x = 0, \qquad x(0) = 0, \quad x'(0) = -2.$$

The auxiliary equation is

$$0 = 2m^2 + 20m + 18 = 2(m+1)(m+9).$$

Since $m = -1$ and $m = -9$, a general solution of the differential equation is

$$x(t) = C_1e^{-t} + C_2e^{-9t}.$$

The initial conditions require

$$0 = x(0) = C_1 + C_2, \qquad -2 = x'(0) = -C_1 - 9C_2.$$

These give $C_2 = -1/4$ and $C_1 = 1/4$, and therefore

$$x(t) = \frac{1}{4}(e^{-9t} - e^{-t}) \text{ m}.$$

(b) The motion is overdamped.

(c) Maximum displacement is achieved when

$$\begin{aligned} 0 = x'(t) &= \frac{1}{4}(-9e^{-9t} + e^{-t}) \\ 9e^{-9t} &= e^{-t} \\ e^{8t} &= 9 \\ t &= \frac{1}{8}\ln 9 \end{aligned}$$

Maximum distance from equilibrium is therefore

$$\left|x\left(\frac{1}{8}\ln 9\right)\right| = \frac{1}{4}\left|e^{-(9/8)\ln 9} - e^{-(1/8)\ln 9}\right| \text{ m}.$$