MATH2132 Test 3 Solutions

Values

8 **1.** Find, in explicit form, a 1-parameter family of solutions of the differential equation

$$\frac{dy}{dx} = 2xy + x.$$

The differential equation is separable,

$$\frac{1}{2y+1}dy = x\,dx.$$

A one-parameter family of solutions is defined implicitly by

$$\begin{aligned}
\frac{1}{2}\ln|2y+1| &= \frac{x^2}{2} + C\\
\ln|2y+1| &= x^2 + 2C\\
|2y+1| &= e^{x^2+2C}\\
2y+1 &= \pm e^{2C}e^{x^2} = De^{x^2}\\
y(x) &= \frac{1}{2}(De^{x^2} - 1)
\end{aligned}$$

8 **2.** (a) Find a 1-parameter family of solutions of the differential equation

$$x^2\frac{dy}{dx} - 3y = 4.$$

(b) Can there be any singular solutions to your family of solutions in part (a)? Explain.

(a) The differential equation is linear, first-order

$$\frac{dy}{dx} - \frac{3}{x^2}y = \frac{4}{x^2}.$$

An integrating factor is $e^{\int(-3/x^2)\,dx} = e^{3/x}$. When we multiply the differential equation by this factor,

$$\begin{aligned}
e^{3/x}\frac{dy}{dx} - \frac{3}{x^2}e^{3/x}y &= \frac{4}{x^2}e^{3/x}\\
\frac{d}{dx}(ye^{3/x}) &= \frac{4}{x^2}e^{3/x}
\end{aligned}$$

Integration gives

$$ye^{3/x} = -\frac{4}{3}e^{3/x} + C \qquad \Longrightarrow \qquad y(x) = -\frac{4}{3} + Ce^{-3/x}.$$

(b) Since the equation is linear, the solution is general, and there can be no singular solutions.

14 **3.** Find a general solution of the differential equation

$$3y''' + 5y'' + 4y' - 2y = 2e^x + x.$$

The auxiliary equation is

$$0 = 3m^3 + 5m^2 + 4m - 2 = (3m - 1)(m^2 + 2m + 2)$$

with solutions

$$m = 1/3, \qquad m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 e^{x/3} + e^{-x}(C_2 \cos x + C_3 \sin x).$$

If we assume a particular solution $y_p(x) = Ae^x + Bx + C$, then

$$3(Ae^x) + 5(Ae^x) + 4(Ae^x + B) - 2(Ae^x + Bx + C) = 2e^x + x.$$

Equating coefficients gives:

$$\begin{aligned} e^x &: 3A + 5A + 4A - 2A = 2 & \Longrightarrow \quad & A = 1/5 \\ x &: -2B = 1 & \Longrightarrow \quad & B = -1/2 \\ 1 &: 4B - 2C = 0 & \Longrightarrow \quad & C = -1 \end{aligned}$$

Thus, $y_p(x) = (1/5)e^x - x/2 - 1$, and

$$y(x) = C_1 e^{x/3} + e^{-x}(C_2 \cos x + C_3 \sin x) + \frac{1}{5}e^x - \frac{x}{2} - 1.$$

10 **4.** (a) A 500 gram mass is suspended from a spring with constant 50 newtons per metre. It is set into motion by releasing it from a position 10 centimetres above its equilibrium position. If a damping force proportional to velocity with coefficient $\beta = 2$ acts on the mass, find its position as a function of time.

(b) If your solution is expressed in the form $Ae^{-\beta t}\sin(\omega t + \phi)$, where A and ω are constants, what is A?

(a) The initial-value problem for displacement $x(t)$ is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 50x = 0, \qquad x(0) = \frac{1}{10}, \quad x'(0) = 0.$$

The auxiliary equation is

$$m^2 + 4m + 100 = 0 \qquad \Longrightarrow \qquad m = \frac{-4 \pm \sqrt{16 - 400}}{2} = -2 \pm 4\sqrt{6}i.$$

Thus,

$$x(t) = e^{-2t}(C_1 \cos 4\sqrt{6}t + C_2 \sin 4\sqrt{6}t).$$

The initial conditions require

$$\frac{1}{10} = C_1, \qquad 0 = -2C_1 + 4\sqrt{6}C_2 \quad \Longrightarrow \quad C_2 = \frac{1}{20\sqrt{6}}.$$

Thus,

$$x(t) = e^{-2t}\left(\frac{1}{10}\cos 4\sqrt{6}t + \frac{1}{20\sqrt{6}}\sin 4\sqrt{6}t\right) \text{ m}.$$

(b) If we write

$$Ae^{-2t}\sin(\omega t + \phi) = e^{-2t}\left(\frac{1}{10}\cos 4\sqrt{6}t + \frac{1}{20\sqrt{6}}\sin 4\sqrt{6}t\right)$$

then, $\omega = 4\sqrt{6}$, and

$$A = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{20\sqrt{6}}\right)^2}.$$