CHAPTER 5 APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

In Chapter 3 we saw that a single differential equation can model many different situations. The linear second-order differential equation, to which we paid so much attention in Chapter 4, represents so many applications, it is undoubtedly the most widely used differential equation in applied mathematics, the physical sciences, and engineering. We concentrate on one of these applications in the first three sections of this chapter, vibrations of masses on the ends of springs. Simplistic as this situation may seem, it serves as a beginning model for many complicated vibrating systems. In addition, whatever results we discover about vibrating mass-spring systems can be interpreted in other physical systems described by linear second-order differential equations. One such is LCR circuits.

5.1 Vibrating Mass-Spring Systems

Consider the situation in Figure 5.1 of a spring attached to a solid wall on one end and a mass of M kilograms on the other. If the mass is set into horizontal motion along the axis of the spring it will continue to do so for some time. The nature of the motion depends on a number of factors such as the tightness of the spring, the amount of mass, whether there is friction between the mass and the surface along which it slides, whether there is friction between the mass and the medium in which it slides, and whether there are any other forces acting on the mass. In this and the next two sections we model this situation with linear second-order differential equations.

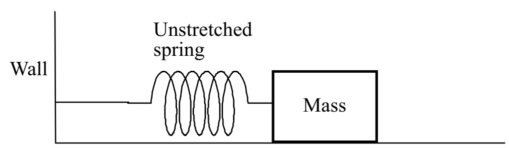


Figure 5.1

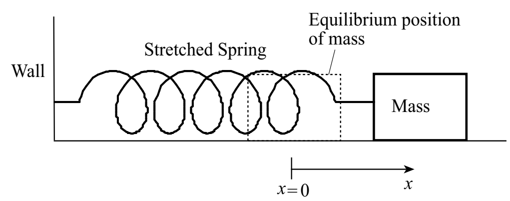


Figure 5.2

Our objective is to predict the position of the mass at any given time, knowing the forces acting on the mass, and how motion is initiated. We begin by establishing a means by which to identify the position of the mass. Most convenient is to let x (in metres) represent the position of the mass relative to the position that it would occupy were the spring unstretched and uncompressed, called the **equilibrium position** (Figure 5.2). We shall then look for x as a function of time t, taking $t = 0$ at the instant that motion is initiated. To determine the differential equation describing oscillations of the mass, we analyze the forces acting on the mass when it is at position x. First there is the spring. Hooke's Law states that when a spring is stretched, the force exerted by the spring in an attempt to restore itself to an unstretched position is proportional to the amount of stretch in the spring. Since x not only identifies the position of the mass, but also represents the stretch in the spring, it follows that the restoring force exerted by the spring on the mass at position x is $-kx$, where $k > 0$ is the constant of proportionality, called the **spring constant**, with units of Newtons per metre. The negative sign indicates that the force is to the left when x is positive and the spring is stretched. In a compressed situation, the spring force should be positive (to the right). This is indeed the case, because with compression, x is negative, and therefore $-kx$ is positive.

In many vibration problems, there is a **damping force**, a force opposing motion that has magnitude directly proportional to the velocity of the mass. It might be a result of air friction with the mass, or it might be due to a mechanical device like a shock absorber on a car, or a combination of such forces. Damping forces are modelled by what is called a **dashpot**; it is shown in (Figure 5.3). Because damping forces are proportional to velocity, and velocity is given by dx/dt, they can be represented in the form $-\beta(dx/dt)$, where $\beta > 0$ is a constant. The negative sign accounts for the fact that damping forces oppose motion; they are in the opposite direction to velocity.

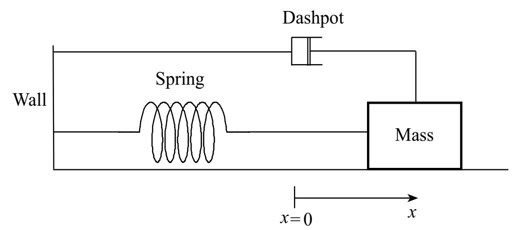


Figure 5.3

There could be other forces that act on the mass; they could depend on both the position of the mass and time. In the event that they depend only on time, we denote them by $F(t)$. (If they depend on position, the resulting differential equation could be nonlinear.) The total force acting on the mass is therefore $-kx - \beta(dx/dt) + F(t)$. According to Newton's second law, this force is equal to mass M times acceleration (at least when M is constant). In addition, we are assuming that the spring itself is so light, in comparison to M, that it can be ignored in the formulation of Newton's second law. Since acceleration is the second derivative of the displacement or position function, d^2x/dt^2, we can write that

$$-kx - \beta\frac{dx}{dt} + F(t) = M\frac{d^2x}{dt^2}. \tag{5.1}$$

When this equation is rearranged into the form,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t), \tag{5.2}$$

we have a linear, second-order differential equation for the position function $x(t)$. The equation is homogeneous or nonhomogeneous depending on whether forces other than the spring and damping act on the mass. In this section and the next, we consider situations in which these are the only two forces acting on the mass; the presence of periodic forces in Section 5.3 leads to the extremely important concept of resonance.

It is perhaps worthwhile pointing out that nonhomogeneity $F(t)$ can arise in other ways, besides the application of other forces on the mass. Suppose, for instance, that the left end of the spring is attached to a support that is not stationary, but moves so that its position, relative to its position when the mass is at its equilibrium position and the spring is unstretched, is $z = f(t)$ (Figure 5.4). This does not affect the damping force (since it is a function of velocity, not position), but it does affect the spring force.

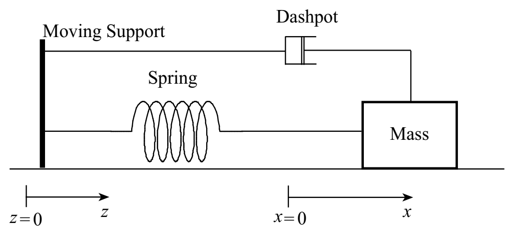


Figure 5.4

When the mass is at position x relative to its equilibrium position, the stretch in the spring is $x - z$. Newton's second law now gives

$$M\frac{d^2x}{dt^2} = -k(x-z) - \beta\frac{dx}{dt}, \tag{5.3}$$

from which

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = kf(t). \tag{5.4}$$

Except for the k factor on the right, this is equation 5.2, but in equation 5.4, $f(t)$ is not a force.

Before considering specific situations, we show that when masses are suspended vertically from springs, their motion is also governed by equation 5.2. To describe the position of the mass M in Figure 5.5 as a function of time t, we need a vertical coordinate system.

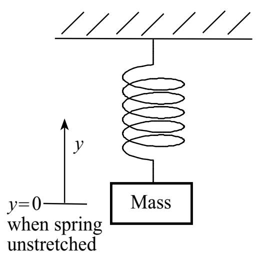


Figure 5.5

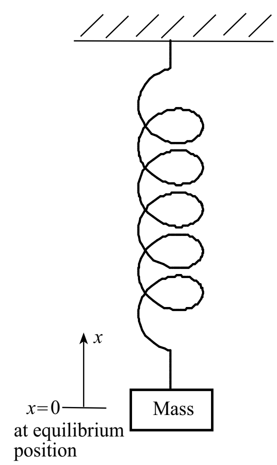


Figure 5.6

There are two natural places to choose the origin. One is to choose $y = 0$ at the position of M when the spring is unstretched. Suppose we do this and choose y as positive upward. When M is a distance y away from the origin, the restoring force of the spring is $-ky$. In addition, if $g = 9.81$ is the acceleration due to gravity, then the force of gravity on M is $-Mg$. In the presence of damping forces or a dashpot, there is a force of the form $-\beta(dy/dt)$, where β is a positive constant. If $F(t)$ represents all other forces acting on M, then the total force on M is $-ky - Mg - \beta(dy/dt) + F(t)$, and Newton's second law for the acceleration of M gives

$$-ky - Mg - \beta\frac{dy}{dt} + F(t) = M\frac{d^2y}{dt^2}.$$

Consequently, the differential equation that determines the position $y(t)$ of M relative to its position when the spring is the unstretched is

$$M\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = -Mg + F(t). \tag{5.5}$$

An alternative for describing vertical oscillations is to attach M to the spring and slowly lower M until it reaches an equilibrium position. At this position, the restoring force of the spring is exactly equal to the force of gravity on the mass, and the mass, left by itself, remains motionless. If $s > 0$ is the amount of stretch in the spring at equilibrium, and $g = 9.81$ is the acceleration due to gravity, then at equilibrium

$$ks - Mg = 0. \tag{5.6}$$

Suppose we take the equilibrium position as $x = 0$ and x as positive upward (Figure 5.6). When M is a distance x away from its equilibrium position, the restoring force of the spring on M is $k(s-x)$. The force of gravity remains as $-Mg$, and the damping force is $-\beta(dx/dt)$. If $F(t)$ accounts for any other forces acting on M, Newton's second law implies that

$$M\frac{d^2x}{dt^2} = k(s-x) - Mg - \beta\frac{dx}{dt} + F(t),$$

or,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = -Mg + ks + F(t).$$

But according to equation 5.6, $ks - Mg = 0$, and hence

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t). \tag{5.7}$$

This differential equation describes displacement $x(t)$ of M relative to the equilibrium position of M.

Equations 5.5 and 5.7 are both linear second-order differential equations with constant coefficients. The advantage of equation 5.7 is that nonhomogeneity $-Mg$ is absent, and this is simply due to a convenient choice of dependent variable (x as opposed to y). Physically, there are two parts to the spring force $k(s-x)$; a part ks and a part $-kx$. Gravity is always acting on M, and that part ks of the spring force is counteracting gravity in an attempt to restore the spring to its unstretched position. Because these forces always cancel, we might just as well eliminate both of them from our discussion. This leaves $-kx$, and we therefore interpret $-kx$ as the spring force attempting to restore the mass to its equilibrium position.

If we choose equation 5.7 to describe vertical oscillations (and this equation is usually chosen over 5.5), we must remember three things: x is measured from equilibrium, $-kx$ is the spring force attempting to restore M to its equilibrium position, and gravity has been taken into account.

Equation 5.7 for vertical oscillations and equation 5.2 for horizontal oscillations are identical; we have the same differential equation describing both types of oscillations. In both cases, x measures the distance of the mass from its equilibrium position. In the horizontal case, this is from the position of the mass when the spring is unstretched; in the vertical case, this is from the position of the mass where it hangs motionless at the end of the spring.

There are three basic ways to initiate motion. First, we can move the mass away from its equilibrium position and then release it, giving it a nonzero initial displacement but zero initial velocity. Secondly, we can strike the mass at the equilibrium position, imparting a nonzero initial velocity but zero initial displacement. And finally, we can give the mass both a nonzero initial displacement and a nonzero initial velocity. Each of these methods adds two initial conditions to the differential equation.

Undamped, Unforced Vibrations

In the remainder of this section, we begin our studies with undamped ($\beta = 0$), unforced ($F(t) = 0$) vibrations. We begin with two numerical examples, and finish with a general discussion.

Example 5.1 A 2-kilogram mass is suspended vertically from a spring with constant 32 newtons per metre. The mass is raised 10 centimetres above its equilibrium position and then released. If damping is ignored, find the position of the mass as a function of time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, differential equation 5.7 for the displacement $x(t)$ of the mass becomes

$$2\frac{d^2x}{dt^2} + 32x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 16x = 0,$$

along with the initial conditions $x(0) = 1/10$, $x'(0) = 0$. The auxiliary equation $m^2 + 16 = 0$ has solutions $m = \pm 4i$. Consequently,

$$x(t) = C_1 \cos 4t + C_2 \sin 4t.$$

The initial conditions require

$$1/10 = x(0) = C_1, \quad 0 = x'(0) = 4C_2.$$

Thus,

$$x(t) = \frac{1}{10}\cos 4t \text{ m}.$$

The graph of this function in Figure 5.7 illustrates that the mass oscillates about its equilibrium position forever. This is a direct result of the fact that damping has been ignored. The mass oscillates up and down from a position 10 cm above the equilibrium position to a position 10 cm below the equilibrium position. We call 10 cm the **amplitude** of the oscillations. It takes $2\pi/4 = \pi/2$ seconds to complete one full oscillation, and we call this the **period** of the oscillations. The **frequency** of the oscillations is the number of oscillations that take place each second and this is the reciprocal of the period, namely $2/\pi$ Hz (hertz). Oscillations of this kind are called **simple harmonic motion**.●

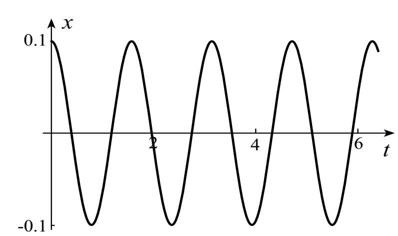


Figure 5.7

The spring in this example might be called "loose". We can see this from equation 5.6. Substitution of $M = 2$, $k = 32$, and $g = 9.81$ gives $s = 0.61$ metres; that is, with a 2 kilogram mass suspended at rest from the spring there is a stretch of 61 centimetres. The period of oscillations $\pi/2$ is quite long and the frequency of oscillations is small $2/\pi$. Contrast this with the much stiffer spring in the following example.

Example 5.2 The 2-kilogram mass in Example 5.1 is suspended vertically from a spring with constant 3200 newtons per metre. The mass is raised 10 centimetres above its equilibrium position and given an initial velocity of 2 metres per second downward. If damping is ignored, find the position of the mass as a function of time.

Solution The differential equation governing motion is

$$2\frac{d^2x}{dt^2} + 3200x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 1600x = 0,$$

along with the initial conditions $x(0) = 1/10$, $x'(0) = -2$. The auxiliary equation $m^2 + 1600 = 0$ has solutions $m = \pm 40i$. Consequently,

$$x(t) = C_1 \cos 40t + C_2 \sin 40t.$$

The initial conditions require

$$1/10 = x(0) = C_1, \quad -2 = x'(0) = 40C_2.$$

Thus,

$$x(t) = \frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t \text{ m}.$$

It is often more convenient to express this function in the form $A\sin(40t + \phi)$. To find A and ϕ, we set

$$\frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t = A\sin(40t + \phi) = A[\sin 40t\,\cos\phi + \cos 40t\,\sin\phi].$$

Because $\sin 40t$ and $\cos 40t$ are linearly independent functions, we equate coefficients to obtain

$$\frac{1}{10} = A\sin\phi, \qquad \frac{-1}{20} = A\cos\phi.$$

When these are squared and added,

$$\frac{1}{100} + \frac{1}{400} = A^2 \qquad \Longrightarrow \qquad A = \frac{\sqrt{5}}{20}.$$

It now follows that ϕ must satisfy the equations

$$\frac{1}{10} = \frac{\sqrt{5}}{20}\sin\phi, \qquad \frac{-1}{20} = \frac{\sqrt{5}}{20}\cos\phi.$$

The smallest positive angle satisfying these is $\phi = 2.03$ radians. The position function of the mass is therefore

$$x(t) = \frac{\sqrt{5}}{20}\sin(40t + 2.03) \text{ m},$$

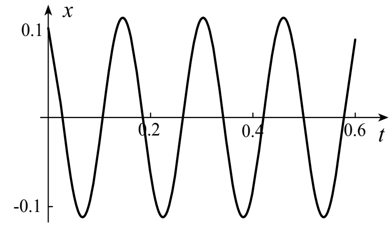


Figure 5.8

a graph of which is shown in Figure 5.8. The amplitude $\sqrt{5}/20$ of the oscillations is slightly larger than that in Example 5.1 due to the fact that the mass was given not only an initial displacement of 10 cm, but also an initial velocity. The spring, with constant $k = 3200$, is one hundred times as tight as that in Example 5.1. The result is a period $\pi/20$ s for the oscillations, one-tenth that in Example 5.1, and ten times as many oscillations per second (frequency is $20/\pi$ Hz).•

General Discussion of Undamped, Unforced Oscillations

When vibrations of a mass M attached to a spring with constant k are unforced and undamped, the differential equation describing displacements $x(t)$ of the mass relative to its equilibrium position is

$$M\frac{d^2x}{dt^2} + kx = 0. \tag{5.8}$$

Because the auxiliary equation $Mm^2 + k = 0$ has solutions $m = \pm\sqrt{k/M}i$, a general solution of the differential equation is

$$x(t) = C_1 \cos\sqrt{\frac{k}{M}}t + C_2 \sin\sqrt{\frac{k}{M}}t. \tag{5.9}$$

This is simple harmonic motion that once again we prefer to write in the form

$$x(t) = A\sin\left(\sqrt{\frac{k}{M}}t + \phi\right), \tag{5.10a}$$

where the amplitude is given by

$$A = \sqrt{C_1^2 + C_2^2}, \tag{5.10b}$$

and angle ϕ is defined by the equations

$$\sin\phi = \frac{C_1}{A} \qquad \text{and} \qquad \cos\phi = \frac{C_2}{A}. \tag{5.10c}$$

Quantity $\sqrt{k/M}$, often denoted by ω, is called the **angular frequency** for the motion. When divided by 2π, $\dfrac{\omega}{2\pi} = \dfrac{\sqrt{k/M}}{2\pi}$, is the **frequency** of the oscillations, the number of oscillations that the mass makes each second. Its inverse $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{k/M}}$ is the **period** of the oscillations, the length of time for the mass to make one complete oscillation. Each of these quantities depends only on the mass M and the spring constant k, not on the initial displacement nor the initial velocity of the mass. Notice that frequency increases with k, indicating that stiffer springs produce faster oscillations. Frequency decreases with M so that heavier masses oscillate more slowly than lighter ones.

Initial conditions enter the calculation of the amplitude of the oscillations and angle ϕ. For instance, if the initial displacement and velocity (at time $t = 0$) are x_0 and v_0, then C_1 and C_2 must satisfy the equations

$$x_0 = C_1 \qquad \text{and} \qquad v_0 = \sqrt{\frac{k}{M}}C_2 = \omega C_2.$$

With these initial conditions,

$$x(t) = A\sin(\omega t + \phi), \tag{5.11a}$$

where

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \tag{5.11b}$$

and ϕ is given by

$$\sin\phi = \frac{x_0}{A} \qquad \text{and} \qquad \cos\phi = \frac{v_0/\omega}{A}. \tag{5.11c}$$

The amplitude of the oscillations is constant because without damping, there is no release of the initial energy supplied to the system with the initial displacement and velocity. Angle ϕ is often called the **phase angle** or **angular phase shift**. Quantity $-\phi/\omega$ is called the **phase shift** as it represents the shift in time of the graph of $A\sin(\omega t + \phi)$ along the t-axis relative to that of $A\sin\omega t$.

Alternative forms for solution 5.9 are discussed in Exercise 16.

The Oscillating Pendulum

Figure 5.9 shows a mass of M kilograms attached to a rod of length L metres, the weight of which we neglect. If the rod is pulled away from the vertical and released, it will oscillate for some time. The displacement of the mass can be measured by the angle that it makes with the vertical. To find θ as a function of time t, we once again use Newton's second law. If damping is ignored, only gravity acts on M, and the component of its weight along the circular arc followed by M is $Mg\sin\theta$. If s measures arc length, counterclockwise from the vertical along the arc followed by M, then d^2s/dt^2 is the acceleration of M, and Newton's second law requires

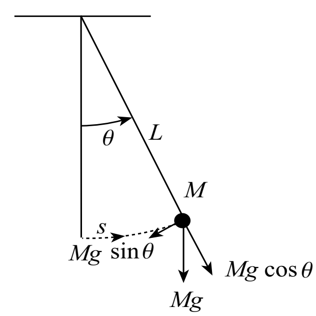


Figure 5.9

$$M\frac{d^2s}{dt^2} = -Mg\sin\theta.$$

We remove M from the equation, and substitute $s = L\theta$,

$$\frac{d^2(L\theta)}{dt^2} = -g\sin\theta.$$

Thus, $\theta(t)$ must satisfy the second-order nonlinear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0. \tag{5.12}$$

In general, this equation is unsolvable, but if we restrict discussions to small displacements, then $\sin\theta$ can be approximated by θ, itself, and equation 5.12 is replaced by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g\theta}{L} = 0. \tag{5.13}$$

Since the auxiliary equation is $m^2 + g/L = 0$, with roots $m = \pm\sqrt{g/L}i$, a general solution of the differential equation is

$$\theta(t) = C_1 \cos\sqrt{\frac{g}{L}}t + C_2 \sin\sqrt{\frac{g}{L}}t.$$

Thus, small displacements of a pendulum are approximated by simple harmonic motion.

EXERCISES 5.1

1. Express the solution in Example 5.2 in the form $x(t) = A\cos(40t - \phi)$.

2. A 1-kilogram mass is suspended vertically from a spring with constant 16 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and then released. Find the position of the mass, relative to its equilibrium position, at any time if damping is ignored.

3. A 100-gram mass is attached to a spring with constant 100 newtons per metre as in Figure 5.2. The mass is pulled 5 centimetres to the right and released. Find the position of the mass if damping, and friction over the sliding surface, are ignored. Sketch a graph of the position function identifying the amplitude, period, and frequency of the oscillations.

4. Repeat Exercise 3 if motion is initiated by striking the mass, at equilibrium, so as to impart a velocity of 3 metres per second to the left.

5. Repeat Exercise 3 if motion is initiated by pulling the mass 5 centimetres to the right and giving it an initial velocity 3 metres per second to the left.

6. Repeat Exercise 3 if motion is initiated by pulling the mass 5 centimetres to the left and giving it an initial velocity 3 metres per second to the left.

7. (a) A 2-kilogram mass is suspended from a spring with constant 1000 newtons per metre. If the mass is pulled 3 centimetres below its equilibrium position and given a downward velocity of 2 metres per second, find its position thereafter. Sketch a graph of the position function identifying the amplitude, period, and frequency of the oscillations.
(b) Do the initial displacement and velocity affect the amplitude, period, and/or frequency?

8. If the mass in Exercise 7 is quadrupled, how does this affect the period and frequency of the oscillations?

9. If the spring constant in Exercise 7 is quadrupled, how does this affect the period and frequency of the oscillations?

10. When a 2-kilogram mass is set into vertical vibrations on the end of a spring, 3 full oscillations occur each second. What is the spring constant if there is no damping?

11. A mass M is suspended from a spring with constant k. Oscillations are initiated by giving the mass a displacement x_0 and velocity v_0. Show that the position of the mass relative to its equilibrium position, when damping is ignored, can be expressed in the form

$$x(t) = A\,\sin\left(\sqrt{k/M}t + \phi\right),$$

where the amplitude is $A = \sqrt{x_0^2 + Mv_0^2/k}$, and ϕ satisfies

$$\sin\phi = \frac{x_0}{A}, \qquad \cos\phi = \frac{\sqrt{M/k}v_0}{A}.$$

12. Use the result of Exercise 11 to show that when the mass on the end of a spring is doubled, the period increases by a factor of $\sqrt{2}$ and the frequency decreases by a factor of $1/\sqrt{2}$.

13. Show the following for oscillations of a mass on the end of a spring when damping is ignored:
(a) Maximum velocity occurs when the mass passes through its equilibrium position. What is the acceleration at this instant?
(b) Maximum acceleration occurs when the mass is at its maximum distance from equilibrium. What is the velocity there?

14. When a spring is suspended vertically, its own weight causes it to stretch. Would this have any effect on our analysis of motion of a mass suspended from the spring?

15. A 100-gram mass is suspended vertically from a spring with constant 40 newtons per metre. The mass is pulled 2 centimetres below its equilibrium position and given an upward velocity of 10 metres per second. Determine:
(a) the position of the mass as a function of time
(b) the amplitude, period, and frequency of the oscillations
(c) all times when the mass has velocity zero
(d) all times when the mass passes through the equilibrium position
(e) all times when the mass is 1 centimetre above the equilibrium position
(f) whether the mass ever has velocity 12 metres per second
(g) the second time the mass is at a maximum height above the equilibrium position.

16. Simple harmonic motion as represented by equation 5.9 can be expressed in alternative forms to 5.10a, namely, $A\sin(\omega t - \phi)$, $A\cos(\omega t + \phi)$, and $A\cos(\omega t - \phi)$. In each case, formula 5.10b for amplitude A is unchanged, only equations 5.10c for angle ϕ change. Show that:
(a) For $A\sin(\omega t - \phi)$,

$$\sin\phi = -\frac{C_1}{A}, \qquad \cos\phi = \frac{C_2}{A}.$$

(b) For $A\cos(\omega t + \phi)$,

$$\sin\phi = -\frac{C_2}{A}, \qquad \cos\phi = \frac{C_1}{A}.$$

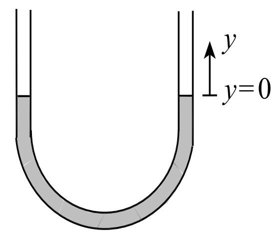
(c) For $A\cos(\omega t - \phi)$,

$$\sin\phi = \frac{C_2}{A}, \qquad \cos\phi = \frac{C_1}{A}.$$

17. At time $t = 0$, a mass M is attached to the end of a hanging spring with constant k, and then released. Assuming that damping is negligible, find the subsequent displacement of the mass as a function of time.

18. A mass of M kilograms is suspended vertically from a spring with constant k newtons per metre. At time $t = 0$, the mass is pulled x_0 metres away from its equilibrium position and given velocity v_0 metres per second. At the same time, the support to which the other end of the spring is attached begins moving up and down with displacement $A \sin \omega t$ metres from its initial position. Find the position of the mass, relative to its equilibrium position, at any time if damping is ignored. Assume that $\omega \neq \sqrt{k/M}$.

19. The figure to the right shows a circular U-tube with radius r metres. Take $y = 0$ as the liquid level in the right tube when both parts of the tube have the same amount of liquid, and let y measure distance upward from this position. If the liquid is disturbed, show that its vertical motion is simple harmonic, and find its period. Take ρ as the density of the liquid.



20. A container of water has mass M kilograms of which m kilograms is water. At time $t = 0$, the container is attached to a spring with constant k newtons per metre, and released. A hole in the bottom of the container allows water to run out at the constant rate of r kilograms per second. If air resistance proportional to velocity acts on the container, set up an initial-value problem for the position of the container while water remains in the container. Do so with the coordinate systems of (a) Figure 5.5, where $y = 0$ is the unstretched position of the spring, and (b) Figure 5.6, where $x = 0$ corresponds to the equilibrium position for a full container. Can you solve these problems?

21. (a) A cube L metres on each side and with mass M kilograms floats half submerged in water. If it is pushed down slightly and then released, oscillations take place. Use Archimedes' principle to find the differential equation governing these oscillations. Assume no damping forces due to the viscosity of the water.
(b) What is the frequency of the oscillations?

22. A cylindrical buoy 20 centimetres in diameter floats partially submerged with its axis vertical. When it is depressed slightly and released, its oscillations have a period equal to 4 seconds. What is the mass of the buoy?

23. A sphere of radius R floats half submerged in water. It is set into vibration by pushing it down slightly and then releasing it. If y denotes the instantaneous distance of its centre below the surface, show that

$$\frac{d^2y}{dt^2} = \frac{-3g}{2R^3}\left(R^2 y - \frac{y^3}{3}\right),$$

where g is the acceleration due to gravity. Is this a linear differential equation?

5.2 Vibrating Mass-Spring Systems With Damping

Damped, Unforced Vibrations

Vibrating mass-spring systems without damping are unrealistic. All vibrations are subject to some degree of damping, and depending on the magnitude of the damping, oscillations either gradually die out, or are completely expunged. Differential equation 5.7 describes the motion of a mass on the end of a spring in the presence of a damping force (with damping constant β) proportional to velocity. When no other forces act on the mass, besides the spring, and gravity for vertical oscillations, the differential equation is homogeneous,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0. \tag{5.14}$$

We shall see that three types of motion can occur called *underdamped*, *critically damped*, and *overdamped*. We illustrate with an example of each before giving a general discussion.

Before we proceed with the examples, it is worthwhile noting that β is most often specified by stating that the damping force is a certain number times the velocity. For example, if we say that the magnitude of the damping force is five times the velocity, then we are saying that $\beta = 5$. We could also say that $\beta = 5$ kilograms per second since these must be the units of β in the mks-system (or SI).

Example 5.3 A 50-gram mass is suspended vertically from a very loose spring with constant 5 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and given velocity 2 metres per second upward. If, during motion, the mass is acted on by a damping force in newtons numerically equal to one-tenth the instantaneous velocity in metres per second, find the position of the mass at any time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $m^2 + 2m + 100 = 0$ has solutions

$$m = \frac{-2 \pm \sqrt{4 - 400}}{2} = -1 \pm 3\sqrt{11}i.$$

Consequently,

$$x(t) = e^{-t}[C_1 \cos(3\sqrt{11}t) + C_2 \sin(3\sqrt{11}t)].$$

The initial conditions require

$$-\frac{1}{20} = x(0) = C_1, \qquad 2 = x'(0) = -C_1 + 3\sqrt{11}C_2,$$

from which $C_2 = 13\sqrt{11}/220$. The position of the mass is therefore given by

$$x(t) = e^{-t}\left[-\frac{1}{20}\cos(3\sqrt{11}t) + \frac{13\sqrt{11}}{220}\sin(3\sqrt{11}t)\right] \text{ m}.$$

The graph of this function in Figure 5.10 clearly indicates how the oscillations decrease in time. This is an example of **underdamped motion**.•

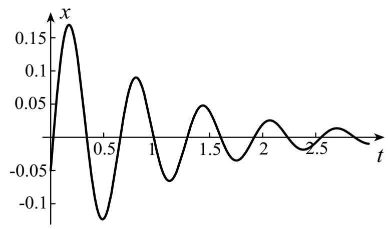


Figure 5.10

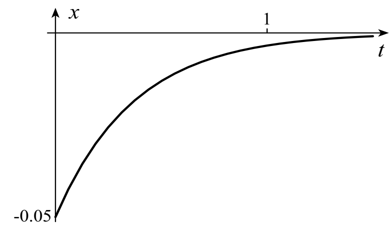


Figure 5.11

Example 5.4 Repeat Example 5.3 if the damping constant is $\beta = 2$.

Solution The differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 100x = 0,$$

along with the same initial conditions. The auxiliary equation $m^2 + 40m + 100 = 0$ has solutions

$$m = \frac{-40 \pm \sqrt{1600 - 400}}{2} = -20 \pm 10\sqrt{3}.$$

Consequently,

$$x(t) = C_1 e^{(-20+10\sqrt{3})t} + C_2 e^{(-20-10\sqrt{3})t}.$$

The initial conditions require

$$-\frac{1}{20} = x(0) = C_1 + C_2, \qquad 2 = x'(0) = (-20 + 10\sqrt{3})C_1 + (-20 - 10\sqrt{3})C_2,$$

from which $C_1 = (2\sqrt{3} - 3)/120$ and $C_2 = -(2\sqrt{3} + 3)/120$. The position of the mass is therefore given by

$$x(t) = \left(\frac{2\sqrt{3} - 3}{120}\right) e^{(-20+10\sqrt{3})t} - \left(\frac{2\sqrt{3} + 3}{120}\right) e^{-(20+10\sqrt{3})t} \text{ m}.$$

The graph of this function is shown in Figure 5.11. This is an example of **overdamped motion**; damping is so large that oscillations are completely eliminated. The mass simply returns to the equilibrium position without passing through it.•

Example 5.5 Repeat Example 5.3 if the damping constant is $\beta = 1$.

Solution The differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $m^2 + 20m + 100 = (m + 10)^2 = 0$ has a repeated solution $m = -10$. Consequently,

$$x(t) = (C_1 + C_2 t)e^{-10t}.$$

The initial conditions require

$$-\frac{1}{20} = x(0) = C_1, \qquad 2 = x'(0) = -10C_1 + C_2,$$

from which $C_2 = 3/2$. The position of the mass is therefore given by

$$x(t) = \left(-\frac{1}{20} + \frac{3t}{2}\right) e^{-10t} \text{ m.}$$

The graph of this function is shown in Figure 5.12. This is an example of **critically damped motion**; any smaller value of the damping constant leads to underdamped motion, and any higher value leads to overdamped motion.•

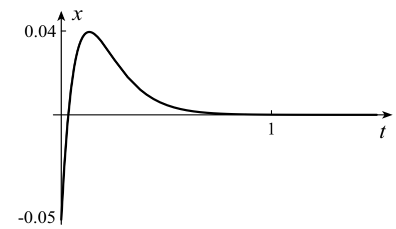


Figure 5.12

General Discussion of Damped, Unforced Motion

We now give a general discussion of differential equation 5.14, clearly delineating values of the parameters M, k, and β that lead to underdamped, critically damped, and overdamped motion. The auxiliary equation associated with

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0 \tag{5.14}$$

is the quadratic equation

$$Mm^2 + \beta m + k = 0, \tag{5.15a}$$

with solutions

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}. \tag{5.15b}$$

Clearly there are three possibilities depending on the value of $\beta^2 - 4kM$.

Underdamped Motion $\quad \beta^2 - 4kM < 0$

When $\beta^2 - 4kM < 0$, roots 5.15b of the auxiliary equation are complex,

$$m = -\frac{\beta}{2M} \pm \frac{\sqrt{4kM - \beta^2}}{2M}i, \tag{5.16}$$

and a general solution of differential equation 5.14 is

$$x(t) = e^{-\beta t/(2M)}\left[C_1 \cos\frac{\sqrt{4kM - \beta^2}}{2M}t + C_2 \sin\frac{\sqrt{4kM - \beta^2}}{2M}t\right]. \tag{5.17}$$

If we set $\omega = \dfrac{\sqrt{4kM - \beta^2}}{2M}$, then

$$x(t) = e^{-\beta t/(2M)}(C_1 \cos\omega t + C_2 \sin\omega t). \tag{5.18}$$

In Section 5.1, we indicated how to express the sine and cosine terms in the form $A\sin(\omega t + \phi)$, so that a simplified expression for underdamped oscillations is

$$x(t) = Ae^{-\beta t/(2M)}\sin(\omega t + \phi). \tag{5.19}$$

The presence of the exponential $e^{-\beta t/(2M)}$ before the trigonometric function indicates that we have oscillations that gradually die out. Except possibly for the starting value and initial slope, a typical graph of this function is shown in Figure 5.13. It is contained between the curves $x = \pm Ae^{-\beta t/(2M)}$. Motion is not periodic, but the time between successive maxima, or between successive minima, or between successive passes through the equilibrium position when going in the same direction, are all the same. This is often called the **quasi-period** of underdamped motion. It is

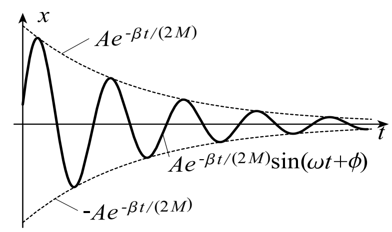


Figure 5.13

$$\frac{2\pi}{\omega} = \frac{2\pi}{\frac{\sqrt{4kM-\beta^2}}{2M}} = \frac{2\pi}{\sqrt{\frac{k}{M} - \frac{\beta^2}{4M^2}}}. \tag{5.20}$$

Since $\dfrac{2\pi}{\sqrt{k/M}}$ is the period of the motion when damping is absent, the quasi-period is larger than this period, but it approaches $\dfrac{2\pi}{\sqrt{k/M}}$ as $\beta \to 0$. Correspondingly, damping decreases the frequency of oscillations. As damping increases and $\beta^2/(4M^2)$ approaches k/M, the quasi-period becomes indefinitely long and oscillations disappear.

Critically Damped Motion $\quad \beta^2 - 4kM = 0$

This is the limiting case of underdamped motion. Roots 5.15b of the auxiliary equation are real and equal $m = -\beta/(2M)$, and a general solution of differential equation 5.14 is

$$x(t) = (C_1 + C_2 t)e^{-\beta t/(2M)}. \tag{5.21}$$

Damping is so large that oscillations are eliminated and the mass returns from its initial position to the equilibrium position passing through the equilibrium position at most once. This situation forms the division between underdamped motion and overdamped motion (yet to come). Any increase of β results in overdamped motion and any decrease results in underdamped oscillations. Except possibly for starting values and initial slopes, typical graphs of this function are shown in Figure 5.14.

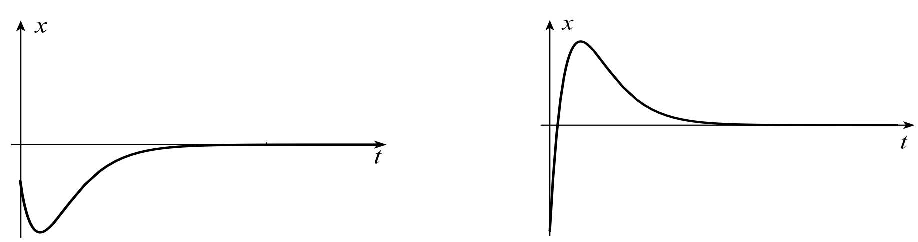


Figure 5.14

Overdamped Motion $\quad \beta^2 - 4kM > 0$

When $\beta^2 - 4kM > 0$, roots 5.15b of the auxiliary equation are real, distinct and negative. A general solution of differential equation 5.14 is

$$x(t) = C_1 e^{(-\beta+\sqrt{\beta^2-4kM})t/(2M)} + C_2 e^{(-\beta-\sqrt{\beta^2-4kM})t/(2M)}. \tag{5.22}$$

Typical graphs of this function are similar to those in Figure 5.14 for critically damped motion.

We now consider further examples of these three possibilities.

Example 5.6 A 100-gram mass is suspended vertically from a spring with constant 5 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and given velocity 2 metres per second upward. If, during motion, the mass is acted on by a damping force in newtons numerically equal to one-twentieth the instantaneous velocity in metres per second, find the position of the mass at any time. Find the quasi-period of the motion.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the differential equation for the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{20}\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad 2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $2m^2 + m + 100 = 0$ has solutions

$$m = \frac{-1 \pm \sqrt{1-800}}{4} = \frac{-1 \pm \sqrt{799}i}{4}.$$

Consequently,

$$x(t) = e^{-t/4}\left[C_1 \cos\left(\frac{\sqrt{799}t}{4}\right) + C_2 \sin\left(\frac{\sqrt{799}t}{4}\right)\right].$$

The initial conditions require

$$-\frac{1}{20} = x(0) = C_1, \qquad 2 = x'(0) = -\frac{C_1}{4} + \frac{\sqrt{799}C_2}{4},$$

from which $C_2 = 159\sqrt{799}/15980$. The position of the mass is therefore given by

$$x(t) = e^{-t/4}\left[-\frac{1}{20}\cos\left(\frac{\sqrt{799}t}{4}\right) + \frac{159\sqrt{799}}{15980}\sin\left(\frac{\sqrt{799}t}{4}\right)\right] \text{ m.}$$

Using the technique suggested in Example 5.2, we can rewrite the displacement in the form

$$x(t) = Ae^{-t/4}\sin\left(\frac{\sqrt{799}t}{4} + \phi\right),$$

where

$$A = \sqrt{\left(-\frac{1}{20}\right)^2 + \left(\frac{159\sqrt{799}}{15980}\right)^2} \approx 0.285661.$$

The graph of these underdamped oscillations is shown in Figure 5.15. Oscillations are bounded by the curves $x = \pm 0.285661e^{-t/4}$, shown dotted. According to formula 5.20, the quasi-period is

$$\sqrt{\frac{5}{1/10)} - \frac{1}{4(400)(25)}} \approx 0.9.\bullet$$

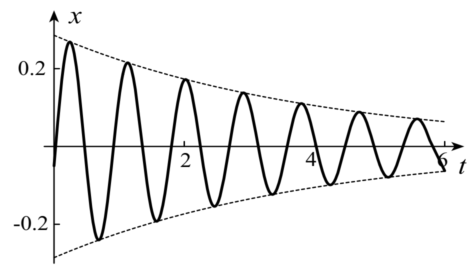


Figure 5.15

Example 5.7 A 4-kilogram mass is attached to a horizontal spring. The mass moves on a frictionless surface, but a dashpot creates a damping force in newtons equal to ten times the velocity of the mass. What spring constant leads to critically damped motion?

Solution Critically damped motion results when spring constant k, mass $M = 4$, and damping factor $\beta = 10$ are related by $\beta^2 - 4kM = 0$; that is, $100 - 4k(4) = 0$. This implies that $k = 25/4$ N/m.•

Example 5.8 A 2-kilogram mass is suspended vertically from a spring with constant 500 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and given velocity 5 metres per second downward. A dashpot is attached to the mass creating a damping force in newtons numerically equal to one hundred times the instantaneous velocity in metres per second. Show that motion of the mass is overdamped and that in 1 second the mass is within 1 millimetre of its equilibrium position.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the initial-value problem for the position $x(t)$ of the mass is

$$2\frac{d^2x}{dt^2} + 100\frac{dx}{dt} + 500x = 0, \qquad x(0) = -\frac{1}{10}, \quad x'(0) = -5.$$

The auxiliary equation $2m^2 + 100m + 500 = 2(m^2 + 50m + 250) = 0$ has solutions

$$m = \frac{-50 \pm \sqrt{2500 - 1000}}{2} = -25 \pm 5\sqrt{15}.$$

With real roots, motion is overdamped and the position function is of the form

$$x(t) = C_1 e^{(-25+5\sqrt{15})t} + C_2 e^{(-25-5\sqrt{15})t}.$$

The initial conditions require

$$-\frac{1}{10} = x(0) = C_1 + C_2, \qquad -5 = x'(0) = (-25 + 5\sqrt{15})C_1 - (25 + 5\sqrt{15})C_2.$$

These can be solved for

$$C_1 = -\frac{\sqrt{15}+1}{20}, \qquad C_2 = \frac{\sqrt{15}-1}{20}.$$

The position of the mass is therefore given by

$$x(t) = -\left(\frac{\sqrt{15}+1}{20}\right)e^{(-25+5\sqrt{15})t} + \left(\frac{\sqrt{15}-1}{20}\right)e^{(-25-5\sqrt{15})t} \text{ m}.$$

If we set $t = 1$, we obtain the position of the mass after one second,

$$x(1) = -\left(\frac{\sqrt{15}+1}{20}\right)e^{(-25+5\sqrt{15})} + \left(\frac{\sqrt{15}-1}{20}\right)e^{(-25-5\sqrt{15})} = -0.000870 \text{ m};$$

that is, the mass is 0.87 millimetres from the equilibrium position.●

EXERCISES 5.2

1. A 1-kilogram mass is suspended vertically from a spring with constant 16 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and then released. Find the position of the mass, relative to its equilibrium position, if a damping force in newtons equal to one-tenth the instantaneous velocity in metres per second acts on the mass.

2. Repeat Exercise 1 if the damping force is equal to ten times the instantaneous velocity.

3. What damping factor creates critically damped motion for the spring and mass in Exercise 1?

4. A 100-gram mass is suspended vertically from a spring with constant 4000 newtons per metre. The mass is pulled 2 centimetres above its equilibrium position and given a downward velocity of 4 metres per second. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to forty times the instantaneous velocity in metres per second. Does the mass ever pass through the equilibrium position?

5. Repeat Exercise 4 if the mass is given a downward velocity of 10 metres per second.

6. (a) A 1-kilogram mass is suspended vertically from a spring with constant 50 newtons per metre. The mass is pulled 5 centimetres above its equilibrium position and given an upward velocity of 3 metres per second. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to fifteen times the instantaneous velocity in metres per second.
(b) Does the mass ever pass through the equilibrium position?
(c) When is the mass 1 centimetre from the equilibrium position?
(d) Sketch a graph of the position function.

7. Repeat Exercise 6 if the initial velocity is 3/4 metre per second downward.

8. Repeat Exercise 6 if the initial velocity is 3 metres per second downward.

9. (a) A 2-kilogram mass is suspended vertically from a spring with constant 200 newtons per metre. The mass is pulled 10 centimetres above its equilibrium position and given an upward velocity of 5 metres per second. Find the position of the mass, relative to its equilibrium position, if a damping force in newtons equal to four times the instantaneous velocity in metres per second also acts on the mass.
(b) What is the maximum distance the mass attains from equilibrium?
(c) When does the mass first pass through the equilibrium position?

10. (a) A 1-kilogram mass is suspended vertically from a spring with constant 40 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and released. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to twice the instantaneous velocity in metres per second. Express the function in the form $Ae^{-at}\sin(\omega t+\phi)$ for appropriate a, A, ω, and ϕ.

(b) Show that the length of time between successive passes through the equilibrium position is constant. What is this time? Twice its value is often called the **quasi period** for overdamped motion? Is it the same as the period of the corresponding undamped system?

11. A mass M is suspended from a spring with constant k. Motion is initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in critically damped motion.

(a) Show that if x_0 and v_0 are both positive or both negative, the mass cannot pass through its equilibrium position.

(b) When x_0 and v_0 have opposite signs, it is possible for the mass to pass through the equilibrium position, but it can do so only once. What condition must x_0 and v_0 satisfy for this to happen?

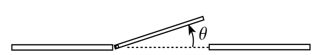
12. A mass M is suspended from a spring with constant k. Motion is initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in overdamped motion.

(a) Show that if x_0 and v_0 are both positive or both negative, the mass cannot pass through its equilibrium position.

(b) When x_0 and v_0 have opposite signs, it is possible for the mass to pass through the equilibrium position, but it can do so only once. What condition must x_0 and v_0 satisfy for this to happen?

13. A weighing platform has weight W and is supported by springs with combined spring constant k. A package with weight w is dropped on the platform so that the two move together. Find a formula for the maximum value of w so that oscillations do not occur. Assume that there is damping in the motion with constant β.

14. In order that a swinging door with moment of inertia I about its hinge return to its closed position, it has a spring attached to it. When the door is open (figure to the right), the spring exerts a force on the door proportional to angle θ with constant of proportionality k. In order to dampen oscillations of the door, there is also a device that exerts a restoring force on the door that is proportional to the angular speed of the door (the constant of proportionality being β). Find values of β that will ensure that the door does not continually oscillate back and forth while closing.



15. Suppose a mass M is attached to a vertical spring with constant k and damping is increased, taking the system from underdamped motion, through critically damped motion, to overdamped motion. Show that the rate at which the mass returns to its equilibrium position is fastest for critically damped motion. Compare rates for underdamped and overdamped motions.

16. A mass M is suspended from a spring with constant k. Oscillations are initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in underdamped motion.

(a) Show that the position of the mass relative to its equilibrium position can be expressed in the form

$$x(t) = Ae^{-\beta t/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t+\phi\right),$$

where A and ϕ are constants.

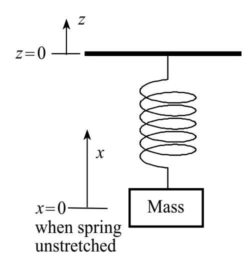
(b) Show that the length of time between successive passes through the equilibrium position is constant. What is this time?

(c) Let t_1, t_2, …, be the times at which the velocity of the mass is equal to zero (and therefore the times at which $x(t)$ has relative maxima and minima. If x_1, x_2, …, are the corresponding values of $x(t)$, show that the ratio

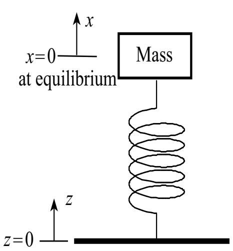
$$\frac{x_n}{x_{n+2}} = e^{2\pi\beta/\sqrt{4kM-\beta^2}},$$

is a constant independent of n. The quantity $2\pi\beta/\sqrt{4kM-\beta^2}$ is called the **logarithmic decrement**.

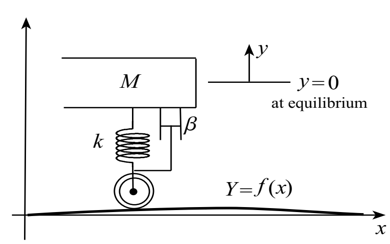
17. At time $t = 0$, a mass M is suspended from a spring with constant k as shown in the figure to the right. The upper end of the spring is attached to a support that is not stationary. Suppose that at time $t = 0$, the support is at a position designated as $z = 0$, and its displacement relative to this position thereafter is given by $z = f(t)$, where z is chosen positive upward. Let the displacement $x(t)$ of the mass be measured relative to its equilibrium position. Assuming that damping is present, determine the differential equation for $x(t)$.



18. At time $t = 0$, a mass M is supported by a spring with constant k as shown in the figure to the right. The lower end of the spring is attached to a support that is not stationary. Suppose that at time $t = 0$, the support is at a position designated as $z = 0$, and its displacement relative to this position thereafter is given by $z = f(t)$, where z is chosen positive upward. Let the displacement $x(t)$ of the mass be measured relative to its equilibrium position. Assuming that damping is present, determine the differential equation for $x(t)$.



19. The figure to the right shows the left front end of a car's suspension system. The road is level for $x < 0$, but has equation $Y = f(x)$ for $x \geq 0$. The speed of the car is a constant v. Assuming that the mass supported by this part of the car is M, and damping is taken into account, find the initial-value problem satisfied by the displacement $y(t)$ of the car from equilibrium. If $f(x)$ is due to a speed bump, it is likely to have a piecewise definition, and handling it by the techniques of Chapter 4 is not particularly convenient. In this situation, Laplace transforms of Chapter 6 are more suitable.



20. Find the differential equation for small displacements of the pendulum in Figure 5.9 when damping proportional velocity is taken into account.

5.3 Vibrating Mass-Spring Systems With External Forces

So far in this chapter we have considered mass-spring systems with damping forces, and in the case of vertical oscillations, gravity is also a consideration. With only these forces, differential equation 5.7 describing motion is homogeneous. Problems become more interesting, and more widely applicable, when other forces are taken into consideration. In particular, periodic forcing functions can lead to *resonance*.

When all other forces acting on the mass in a damped mass-spring system are grouped together into one term denoted by $F(t)$, the differential equation describing motion is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t). \tag{5.23}$$

We consider various possibilities for $F(t)$. To begin with, you may have noticed that in every example of masses sliding along horizontal surfaces (Figure 5.16), we have ignored friction between the mass and the surface. Suppose we now take it into account. If the coefficient of kinetic friction between the mass and surface is μ (see Section 3.2), then the force of friction retarding motion has magnitude μMg where $g > 0$ is the acceleration due to gravity. Entering this force into differential equation 5.23 for all time is a problem due to the difficulty in specifying the direction of the force. Certainly we can say that friction is always in a direction opposite to velocity, and we can represent it in the form $-\mu Mg\dfrac{v}{|v|}$, but entering this into equation 5.23 destroys linearity of the equation. The quotient $-v/|v|$ has values ± 1 depending on whether v is negative or positive; it determines the direction of the frictional force. When v is positive, friction is negative (to the left), and when v is negative, friction is positive (to the right). What this means is that each time the mass changes direction, the differential equation must be reconstituted with the appropriate sign attached to μMg. The following example is an illustration.

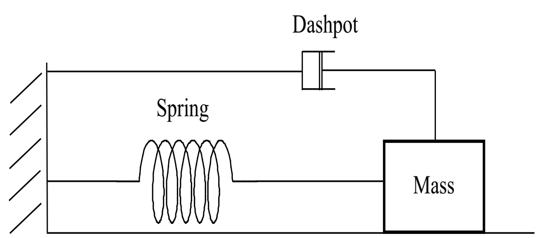


Figure 5.16

Example 5.9 A 1-kilogram mass, attached to a spring with constant 16 newtons per metre, slides horizontally along a surface where the coefficient of kinetic friction between surface and mass is $\mu = 1/10$. Motion is initiated by pulling the mass 10 centimetres to the right of its equilibrium position and giving it velocity 1 metre per second to the left. If any damping forces are negligible, find the point where the mass comes to an instantaneous stop for the second time.

Solution While the mass is travelling to the left for the first time, the force of friction is to the right, and therefore the initial-value problem for its position during this time is

$$\frac{d^2x}{dt^2} + 16x = \left(\frac{1}{10}\right)(1)g, \qquad x(0) = \frac{1}{10}, \quad x'(0) = -1,$$

where $g = 9.81$. Since the auxiliary equation $m^2 + 16 = 0$ has roots $m = \pm 4i$, a general solution of the associated homogeneous differential equation is $x_h(t) = C_1 \cos 4t + C_2 \sin 4t$. It is easy to spot that a particular solution of the nonhomogeneous equation is $x_p(t) = g/160$, and therefore a general solution of the nonhomogeneous differential equation is

$$x(t) = C_1 \cos 4t + C_2 \sin 4t + \frac{g}{160}.$$

The initial conditions require

$$\frac{1}{10} = x(0) = C_1 + \frac{g}{160}, \qquad -1 = x'(0) = 4C_2.$$

Hence,

$$x(t) = \left(\frac{1}{10} - \frac{g}{160}\right) \cos 4t - \frac{1}{4} \sin 4t + \frac{g}{160} \text{ m}.$$

This represents the position of the mass only while it is travelling to the left for the first time. To determine the time and place at which the mass stops moving to the left, we set the velocity equal to zero,

$$0 = \frac{dx}{dt} = -4\left(\frac{1}{10} - \frac{g}{160}\right) \sin 4t - \cos 4t.$$

This equation can be simplified to

$$\tan 4t = \frac{40}{g - 16},$$

solutions of which are

$$t = \frac{1}{4} \operatorname{Tan}^{-1}\left(\frac{40}{g - 16}\right) + \frac{n\pi}{4},$$

where n is an integer. The only acceptable solution is the smallest positive one, and this occurs for $n = 1$, giving $t = 0.431082$ s. The position of the mass at this time is $x(0.431082) = -0.191663$ m. The mass will move from this position if the spring force is sufficient to overcome the force of static friction. Let us suppose that the coefficient of static friction is $\mu_s = 1/5$ (see Section 3.2). This means that the smallest force necessary for the mass to move has magnitude $(1/5)(1)(9.81) = 1.962$ N. Since the spring force at the first stopping position is $0.191663(16) = 3.06661$ N, it is more than enough to overcome the force of static friction.

For the return trip to the right, friction is to the left, and therefore the initial-value problem for position is

$$\frac{d^2x}{dt^2} + 16x = -\frac{g}{10}, \qquad x(0) = -0.191663, \quad x'(0) = 0.$$

For simplicity, we have reinitialized time $t = 0$ to commencement of motion to the right (see Exercise 1 for the analysis without reinitializing time). A general solution of this differential equation is

$$x(t) = C_3 \cos 4t + C_4 \sin 4t - \frac{g}{160}.$$

The initial conditions require

$$-0.191663 = x(0) = C_3 - \frac{g}{160}, \qquad 0 = x'(0) = 4C_4.$$

Thus,

$$x(t) = \left(\frac{g}{160} - 0.191663\right) \cos 4t - \frac{g}{160} \text{ m}.$$

The mass comes to rest when

$$0 = \frac{dx}{dt} = -4\left(\frac{g}{160} - 0.191663\right) \sin 4t,$$

solutions of which are given by $t = n\pi/4$ where n is an integer. The smallest positive value is $t = \pi/4$ and the position of the mass at this time is

$$x(\pi/4) = \left(\frac{g}{160} - 0.191663\right) \cos \pi - \frac{g}{160} = 0.069038 \text{ m};$$

that is, the mass is 6.9 cm to the right of the equilibrium position. The spring force is still sufficient to overcome the force of friction and the mass will again move to the left.•

Periodic Forcing Functions and Resonance

We now consider the application of periodic forcing functions to masses on the ends of springs. When an external force $F \sin \omega t$, where $F > 0$ and $\omega > 0$ are constants, acts on the mass in a mass-spring system, differential equation 5.23 describing motion becomes

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F \sin \omega t. \tag{5.24}$$

We begin discussions with systems that have no damping, somewhat unrealistic perhaps, but essential ideas are not obscured by intensive calculations.

Example 5.10 A mass of M kilograms is suspended from a spring with constant k newtons per metre. It is given initial displacement x_0 metres and initial velocity v_0 metres per second. Assume that damping is negligible during its subsequent motion, but an external force $F \sin \omega t$ newtons acts on the mass. (a) Find the position of the mass as a function of time when $\omega \neq \sqrt{k/M}$. Is it periodic? (b) Discuss the motion of the mass when $\omega = \sqrt{k/M}$.

Solution The initial-value problem for position of the mass is

$$M\frac{d^2x}{dt^2} + kx = F \sin \omega t, \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The auxiliary equation $Mm^2 + k = 0$ has solutions $m = \pm\sqrt{k/M}i$. Suppose we set $\omega_0 = \sqrt{k/M}$. Then, a general solution of the associated homogeneous differential equation is $x_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$.

(a) When $\omega \neq \omega_0$, undetermined coefficients suggests a particular solution of the form $x_p(t) = A \sin \omega t + B \cos \omega t$. Substitution into the differential equation leads to $x_p(t) = [F/(k - M\omega^2)] \sin \omega t$. Thus, a general solution of the nonhomogeneous differential equation is

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F}{k - M\omega^2} \sin \omega t.$$

The initial conditions require

$$x_0 = x(0) = C_1, \qquad v_0 = x'(0) = \omega_0 C_2 + \frac{F\omega}{k - M\omega^2}.$$

Thus, the position of the mass at any time is

$$x(t) = x_0 \cos \omega_0 t + \frac{1}{\omega_0} \left(v_0 - \frac{F\omega}{k - M\omega_2} \right) \sin \omega_0 t + \frac{F}{k - M\omega^2} \sin \omega t.$$

This is the sum of two periodic functions with different periods; the first two terms have period $2\pi/\omega_0$, and the third term has period $2\pi/\omega$. Their sum is periodic if, and only if, ω/ω_0 is a rational number. When the ratio is a rational number, say p/q in lowest terms, then $x(t)$ has period $2\pi q/\omega_0$ or $2\pi p/\omega$.
(b) If $\omega = \omega_0$, we should take the particular solution in the form $x_p(t) = t(A \sin \omega t + B \cos \omega t)$. Substitution into the differential equation leads to $x_p(t) = -\frac{Ft}{2M\omega} \cos \omega t$, and therefore a general solution of the differential equation with nonhomogeneity $F \sin \omega t$ is

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t - \frac{Ft}{2M\omega} \cos \omega t.$$

The initial conditions require

$$x_0 = x(0) = C_1, \qquad v_0 = x'(0) = \omega C_2 - \frac{F}{2M\omega}.$$

Thus, the position of the mass when the forcing function is $F \sin \omega t$ is

$$x(t) = x_0 \cos \omega t + \frac{1}{\omega} \left(v_0 + \frac{F}{2M\omega} \right) \cos \omega t.$$

A graph of this function would look somewhat like that in Figure 5.17. The last term in the solution has led to oscillations that become unbounded. This is a direct result of the fact that when $\omega = \omega_0$, the frequency of the forcing term is equal to the frequency at which the system would oscillate were no forcing term present (the so-called **natural frequency** of the system). (Think of this as similar to a parent pushing a child on a swing. Every other time the swing begins its downward motion, the parent applies a force, resulting in the child going higher and higher. The parent applies the force at the same frequency as the motion of the swing.)•

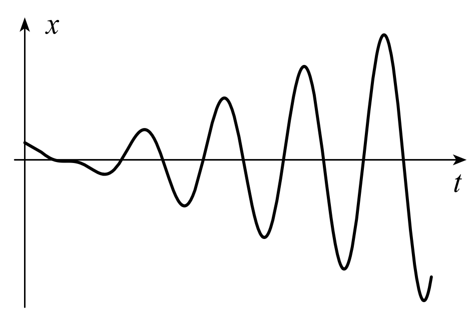


Figure 5.17

This phenomenon of ever increasing oscillations due to a forcing function with the same frequency as the natural frequency of the system is known as **resonance**.

Because the system is undamped, we refer to this as **undamped resonance**.

A natural question at this point is to ask whether periodic forces that are not sinusoidal can cause resonance. We cannot answer this question conveniently now because the techniques of Chapter 4 do not handle periodic, non-sinusoidal forces easily. Laplace transforms of Chapter 6 are exceptional in dealing with such forces, and we will use them to show that non-sinusoidal periodic forces can indeed cause undamped resonance.

Resonance also occurs in damped systems, but there is a difference; oscillations can be large depending on the degree of damping and the forcing frequency, but they cannot become unbounded. Differential equation 5.24 describes motion of a damped mass-spring system in the presence of a periodic forcing function. Equations 5.17–5.21 define general solutions of the associated homogeneous equation, and it is clear that none of these solutions contain the nonhomogeneity $A\sin\omega t$ for any ω. To put it another way, in the presence of damping, simple harmonic motion is not possible, and therefore the system does not have a natural frequency. Resonance as found in undamped systems is therefore not possible. For underdamped motion, however, oscillations can be large, depending on the degree of damping and the frequency of the applied periodic force, and this is again known as resonance, but we call it **damped resonance**. We illustrate in the following example.

Example 5.11 A 1-kilogram mass is at rest, suspended from a spring with constant 65 newtons per metre. Attached to the mass is a dashpot that creates a damping force equal to twice the velocity of the mass whenever the mass is in motion. At time $t = 0$, a vertical force $3\sin\omega t$ begins to act on the mass. Find the position function for the mass. For what value of ω are oscillations largest?

Solution The initial-value problem for the motion of the mass is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 65x = 3\sin\omega t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation $m^2 + 2m + 65 = 0$ has solutions $m = -1 \pm 8i$ so that a general solution of the associated homogeneous differential equation is $x_h(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t)$. A particular solution can be found in the form $x_p(t) = A\sin\omega t + B\cos\omega t$ by undetermined coefficients. The result is

$$x_p(t) = \frac{3(65-\omega^2)}{(65-\omega^2)^2 + 4\omega^2}\sin\omega t - \frac{6\omega}{(65-\omega^2)^2+4\omega^2}\cos\omega t.$$

A general solution of the nonhomogeneous differential equation is therefore

$$x(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t) + \frac{3}{(65-\omega^2)^2+4\omega^2}\left[(65-\omega^2)\sin\omega t - 2\omega\cos\omega t\right].$$

The initial conditions require

$$0 = x(0) = C_1 - \frac{6\omega}{(65-\omega^2)^2+4\omega^2}, \qquad 0 = x'(0) = -C_1 + 8C_2 + \frac{3\omega(65-\omega^2)}{(65-\omega^2)^2+4\omega^2}.$$

These give

$$C_1 = \frac{6\omega}{(65-\omega^2)^2+4\omega^2}, \qquad C_2 = \frac{3\omega(\omega^2-63)}{8[(65-\omega^2)^2+4\omega^2]},$$

and the position of the mass is therefore

$$x(t) = \frac{3\omega e^{-t}}{8[(65-\omega^2)^2+4\omega^2]}[16\cos 8t + (\omega^2-63)\sin 8t] + \frac{3}{(65-\omega^2)^2+4\omega^2}\left[(65-\omega^2)\sin\omega t - 2\omega\cos\omega t\right] \text{ m.}$$

The terms involving $\cos 8t$ and $\sin 8t$ are called the **transient** part of the solution, transient because the e^{-t} factor effectively eliminates these terms after a long time. The terms involving $\sin\omega t$ and $\cos\omega t$, not being subjected to such a factor, do not diminish in time. They are called the **steady-state** part of the solution. In Figure 5.18a we have shown the transient solution; Figure 5.18b shows the steady-state solution with the specific choice $\omega = 4$; and Figure 5.18c shows their sum.

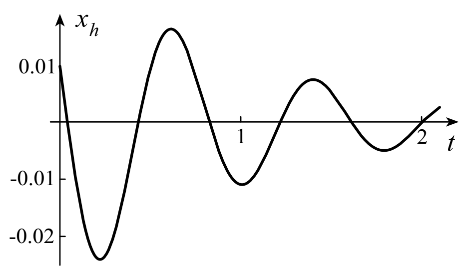


Figure 5.18a

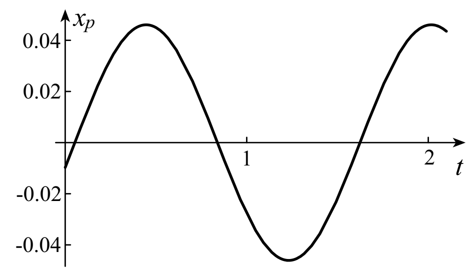


Figure 5.18b

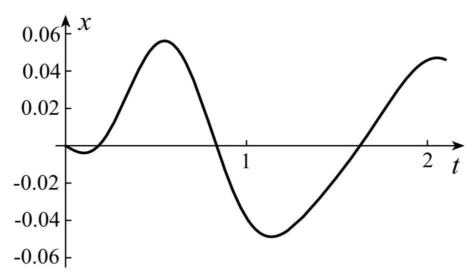


Figure 5.18c

When the forcing frequency is equal to the natural frequency in undamped systems, resonance in the form of unbounded oscillations occurs. Inspection of the above solution indicates that for no value of ω can oscillations become unbounded in this damped system. However, there is a value of ω that makes oscillations largest relative to all other values of ω. In particular, because the transient part of the solution becomes negligible after a sufficiently long time, we are interested in maximizing the amplitude of the steady-state part of the solution. It is the particular solution $x_p(t)$. The amplitude of the oscillations represented by this term is

$$\sqrt{\left[\frac{3(65-\omega^2)}{(65-\omega^2)^2+4\omega^2}\right]^2 + \left[\frac{-6\omega}{(65-\omega^2)^2+4\omega^2}\right]^2} = \frac{3}{\sqrt{(65-\omega^2)^2+4\omega^2}};$$

that is, the steady-state solution can be expressed in the form

$$x_p(t) = \frac{3}{\sqrt{(65-\omega^2)^2+4\omega^2}}\sin(\omega t+\phi)$$

for some ϕ. To maximize the amplitude we minimize $(65-\omega^2)^2+4\omega^2$. Setting its derivative equal to zero gives

$$0 = 2(65-\omega^2)(-2\omega)+8\omega,$$

and the only positive solution of this equation is $\omega = 3\sqrt{7}$. For this value of ω, the steady-state solution becomes

$$x_p(t) = \frac{3}{16}\sin(3\sqrt{7}t).$$

Maximum oscillations have been realized and the system is said to be in damped resonance. We have shown a graph of this function in Figure 5.19. Compare the scale on the vertical axis in this figure to that in Figure 5.18b where $\omega = 4$. We have shown a plot of amplitude versus ω in Figure 5.20.•

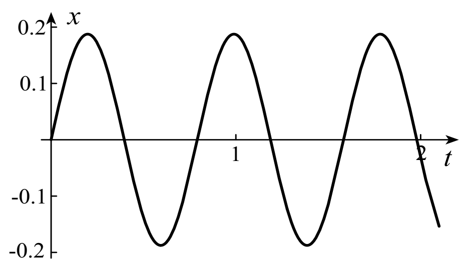


Figure 5.19

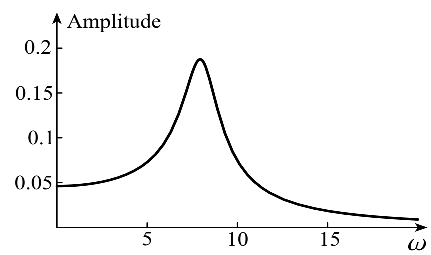


Figure 5.20

We now give a general discussion of damped resonance. When a damped, vibrating mass-spring system is subjected to a sinusoidal input $F\sin\omega t$, the differential equation determining displacements of the mass is

$$M\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx = F\sin\omega t.$$

Because we are assuming that the motion is underdamped, a general solution of the associated homogeneous equation is given by equation 5.17,

$$x_h(t) = e^{-\beta t/(2M)}\left[C_1\cos\frac{\sqrt{4kM-\beta^2}}{2M}t + C_2\sin\frac{\sqrt{4kM-\beta^2}}{2M}t\right].$$

A particular solution can be obtained with undetermined coefficients, assuming the solution in the form

$$x_p(t) = B\sin\omega t + D\cos\omega t.$$

When we substitute this into the differential equation, we get

$$\begin{aligned} M(-\omega^2B\sin\omega t-\omega^2D\cos\omega t)+\beta(\omega B\cos\omega t-\omega D\sin\omega t)& \\ +k(B\sin\omega t+D\cos\omega t) &= F\sin\omega t. \end{aligned}$$

We now equate coefficients of terms in $\sin\omega t$ and $\cos\omega t$,

$$\begin{aligned} -M\omega^2B-\omega\beta D+kB &= F, \\ -M\omega^2D+\omega\beta B+kD &= 0. \end{aligned}$$

The solution of these equations is

$$B = \frac{F(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}, \qquad D = \frac{-\beta\omega F}{(k - M\omega^2)^2 + \beta^2\omega^2}.$$

Thus,

$$x_p(t) = \frac{F(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2} \sin \omega t - \frac{\beta\omega F}{(k - M\omega^2)^2 + \beta^2\omega^2} \cos \omega t.$$

A general solution of the differential equation is $x(t) = x_h(t) + x_p(t)$. We are interested only in the steady-state part of the solution, namely $x_p(t)$. We can write it in the form $x_p(t) = A \sin(\omega t + \phi)$, where the amplitude is given by

$$A = \sqrt{\left[\frac{F(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}\right]^2 + \left[\frac{-\beta\omega F}{(k - M\omega^2)^2 + \beta^2\omega^2}\right]^2},$$

and this simplifies to $A = \dfrac{F}{\sqrt{(k - M\omega^2)^2 + \beta^2\omega^2}}$. Thus,

$$x_p(t) = \frac{F}{\sqrt{(k - M\omega^2)^2 + \beta^2\omega^2}} \sin(\omega t + \phi).$$

The quantity

$$Q(\omega) = \frac{1}{\sqrt{(k - M\omega^2)^2 + \beta^2\omega^2}} \tag{5.25}$$

is called the **gain factor**, or just plain **gain**, with units of metres per newton. It measures the increase in the amplitude of the motion per newton increase of the applied force. For instance, if the gain is 0.01, then the amplitude of the oscillations increases by 1 centimetre for each newton increase of the applied force. It depends on all four physical quantities in the system, M, k, β, and ω. Our primary interest is in how it depends on ω for fixed M, k, and β. But it is also of interest to see its dependence on β for fixed values of M, k, and ω.

Damped resonance occurs when ω is chosen to maximize the amplitude of $x_p(t)$. This occurs when $Q(\omega)$ is maximized (since F is fixed), and this means when $(k - M\omega^2)^2 + \beta^2\omega^2$ is minimized. Critical values of this function are defined by

$$0 = \frac{d}{d\omega}[(k - M\omega^2)^2 + \beta^2\omega^2 = 2(k - M\omega^2)(-2M\omega) + 2\beta^2\omega.$$

The nontrivial solution of this equation is

$$\omega = \sqrt{\frac{k}{M} - \frac{\beta^2}{2M^2}}.$$

This is the applied frequency for damped resonance. The gain at this frequency is

$$\frac{1}{\sqrt{\left(k - k + \dfrac{\beta^2}{2M}\right)^2 + \beta^2\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)}} = \frac{1}{\beta\sqrt{\dfrac{k}{M} - \dfrac{\beta^2}{4M^2}}}.$$

Notice that as β approaches zero, the frequency at damped resonance approaches $\sqrt{k/M}$, the frequency for undamped resonance.

Figure 5.21 shows plots of $Q(\omega)$ for four values of β when $k = 1$ and $M = 1$. As β approaches zero, the gain at damped resonance becomes very large.

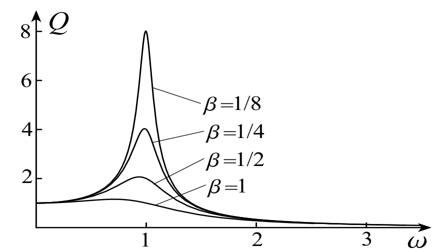


Figure 5.21

Example 5.12 A mass of 500 grams, attached to a very loose spring with constant 2 newtons per metre, is at rest on a horizontal table. The other end of the spring is attached to a block that oscillates back and forth, starting at time $t = 0$ according to $0.01 \sin 3t$ metres. During its subsequent motion, the mass experiences a damping force in newtons that is 1/10 its velocity in metres per second. Find the displacement of the mass from its equilibrium position as a function of time. Determine the amplitude of the steady-state oscillations.

Solution According to equation 5.4, the differential equation describing the displacement of the mass is

$$\frac{1}{2}\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + 2x = 2(0.01 \sin 3t),$$

or,

$$5\frac{d^2x}{dt^2} + \frac{dx}{dt} + 20x = \frac{1}{5}\sin 3t.$$

Initial conditions are $x(0) = x'(0) = 0$. The auxiliary equation is

$$0 = 5m^2 + m + 20, \quad \text{with solutions} \quad m = \frac{-1 \pm \sqrt{1 - 4(5)(20)}}{10} = \frac{-1 \pm \sqrt{399}i}{10}.$$

A general solution of the associated homogeneous equation is therefore

$$x_h(t) = e^{-t/10}\left(C_1 \cos\frac{\sqrt{399}t}{10} + C_2 \sin\frac{\sqrt{399}t}{10}\right).$$

When we substitute a particular solution of the form $x_p(t) = A \sin 3t + B \cos 3t$ into the nonhomogeneous equation, we obtain

$$5(-9A \sin 3t - 9B \cos 3t) + (3A \cos 3t - 3B \sin 3t) + 20(A \sin 3t + B \cos 3t) = \frac{1}{5}\sin 3t.$$

Equating coefficients of $\sin 3t$ and $\cos 3t$ gives the equations

$$-45A - 3B + 20A = \frac{1}{5}, \quad -45B + 3A + 20B = 0.$$

The solution is $A = -1/78$ and $B = 1/390$. Thus, $x_p(t) = -\frac{1}{78}\sin 3t + \frac{1}{390}\cos 3t$, and a general solution of the differential equation is

$$x(t) = e^{-t/10}\left(C_1 \cos\frac{\sqrt{399}t}{10} + C_2 \sin\frac{\sqrt{399}t}{10}\right) - \frac{1}{78}\sin 3t + \frac{1}{390}\cos 3t.$$

The initial conditions require

$$0 = x(0) = C_1 + \frac{1}{390}, \quad 0 = x'(0) = -\frac{C_1}{10} + \frac{\sqrt{399}C_2}{10} - \frac{3}{78}.$$

These give $C_1 = -\dfrac{1}{390}$ and $C_2 = \dfrac{149}{390\sqrt{399}}$. The displacement of the mass is therefore

$$x(t) = e^{-t/10}\left(-\frac{1}{390}\cos\frac{\sqrt{399}t}{10} + \frac{149}{390\sqrt{399}}\sin\frac{\sqrt{399}t}{10}\right) - \frac{1}{78}\sin 3t + \frac{1}{390}\cos 3t \text{ m.}$$

The amplitude of the steady-state oscillations is

$$A = \sqrt{\left(\frac{1}{78}\right)^2 + \left(\frac{1}{390}\right)^2} = \frac{\sqrt{26}}{390} \text{ m.}\bullet$$

Amplitude Modulation

Suppose an oscillatory system is modelled by differential equation 5.2 with no damping and an external forcing function $A\cos\omega t$, where A and ω are positive constants. Provided ω is not equal to the natural frequency $\omega_0 = \sqrt{k/M}$ of the system, it is straightforward to show that a general solution of this equation is

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{k - M\omega^2}\cos\omega t.$$

If the system has no initial energy at time $t = 0$, and the force $A\cos\omega t$ excites the system, then the initial conditions are $x(0) = 0$ and $x'(0) = 0$, and these imply that $C_2 = 0$ and $C_1 = A/(k - M\omega^2)$. Thus,

$$x(t) = \frac{A}{M(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t).$$

By using the trigonometric identity $\cos F - \cos G = -2\sin\left(\dfrac{F+G}{2}\right)\sin\left(\dfrac{F-G}{2}\right)$, we can write this solution in the form

$$x(t) = \frac{2A}{M(\omega^2 - \omega_0^2)}\sin\left(\frac{\omega - \omega_0}{2}\right)t\sin\left(\frac{\omega + \omega_0}{2}\right)t.$$

If ω is very close to ω_0 (they have been assumed not equal), then $|\omega_0 - \omega|$ is very much smaller than $\omega_0 + \omega$. Suppose for instance that $\omega = 1.1\omega_0$, in which case

$$\begin{aligned} x(t) &= \frac{2A}{M(1.1^2\omega_0^2 - \omega_0^2)}\sin\left(\frac{1.1\omega_0 - \omega_0}{2}\right)t\sin\left(\frac{1.1\omega_0 + \omega_0}{2}\right)t \\ &= \frac{9.52A}{M\omega_0^2}\sin(0.05\omega_0 t)\sin(1.05\omega_0 t). \end{aligned}$$

We have plotted this function in Figure 5.22a. We have also included plots of the functions $\pm\dfrac{9.52A}{M\omega_0^2}\sin(0.05\omega_0 t)$ (the dashed curves), and have done so for the following reason. Obviously the period of the dashed curves is much larger than that of $x(t)$. We could look at the function $x(t)$ as the sine function $\sin(1.05\omega_0 t)$ with a time varying amplitude $\pm\dfrac{9.52A}{M\omega_0^2}\sin(0.05\omega_0 t)$. In Figure 5.22b, we have plotted the same curves when $\omega = 1.05\omega_0$, in which case

$$x(t) = \frac{19.5}{M\omega_0^2}\sin(0.025\omega_0 t)\sin(1.025\omega_0 t).$$

The larger frequency $1.025\omega_0$ has not changed much from $1.05\omega_0$, but the smaller frequency has been cut in half. In addition, there is a substantial increase in the time varying amplitude; it has more than doubled. What we are seeing in these graphs are called **beats**. Beats can actually be heard when two musical instruments produce sounds with frequencies that are very close to each other. In electronics, this is called **amplitude modulation**.

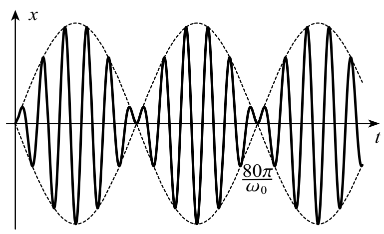


Figure 5.22a

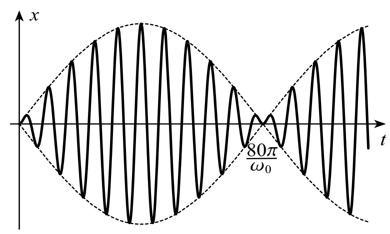


Figure 5.22b

EXERCISES 5.3

1. Repeat Example 5.9 without reinitializing time for movement to the right.

2. A 0.5-kilogram mass sits on a table attached to a spring with constant 18 newtons per metre (Figure 5.16). The mass is pulled so as to stretch the spring 6 centimetres and then released.
(a) If friction between the mass and the table creates a force of 0.5 newtons that opposes motion, but damping is negligible, show that the differential equation determining motion is

$$\frac{d^2x}{dt^2} + 36x = 1, \quad x(0) = 0.06,\ x'(0) = 0.$$

Assume that the coefficient of static friction is twice the coefficient of kinetic friction.
(b) Find where the mass comes to rest for the first time. Will it move from this position?

3. Repeat Exercise 2 given that the mass is pulled 25 centimetres to the right.

4. A 200-gram mass rests on a table attached to an unstretched spring with constant 5 newtons per metre. The mass is given a velocity of 1/2 metre per second to the right. During the subsequent motion, the coefficient of kinetic friction between mass and table is $\mu_k = 1/4$, but damping is negligible. Where does the mass come to a complete stop? Assume that the coefficient of static friction is $\mu_s = 1/2$.

5. Repeat Exercise 4 if the initial velocity is 2 metres per second.

6. A 100-gram mass is suspended from a spring with constant 4000 newtons per metre. At its equilibrium position, it is suddenly (time $t = 0$) given an upward velocity of 10 metres per second. If an external force $3\cos 100t$, $t \geq 0$ acts on the mass, find its displacement as a function of time. Does undamped resonance occur?

7. Repeat Exercise 6 if the external force is $3\cos 200t$.

8. A vertical spring having constant 64 newtons per metre has a 1-kilogram mass attached to it. An external force 2 $\sin 4t$, $t \geq 0$ is applied to the mass. If the mass is at rest at its equilibrium position at time $t = 0$, and damping is negligible, find the position of the mass as a function of time. Does undamped resonance occur?

9. Repeat Exercise 8 if the external force is 2 $\sin 8t$.

10. A mass M, connected to a spring with constant k, sits motionless on a table. The other end of the spring is attached to a movable support. At time $t = 0$, the support moves horizontally with displacement $A \sin \omega t$, thus causing the spring to compress, and the mass to move. Assuming no damping, and no friction between the mass and the surface along which it slides, find the displacement of the mass when (a) $\omega \neq \sqrt{k/M}$, and (b) when $\omega = \sqrt{k/m}$.

11. A mass M is suspended from a vertical spring with constant k. If an external force $F(t) = A \cos \omega t$ is applied to the mass for $t > 0$, find the value of ω that causes undamped resonance.

12. A 200-gram mass suspended vertically from a spring with constant 10 newtons per metre is set into vibration by an external force in newtons given by 4 $\sin 10t$, $t \geq 0$. During the motion a damping force in newtons equal to 3/2 the velocity on the mass in metres per second acts on the mass. Find the position of the mass as a function of time t.

13. (a) A 1-kilogram mass is motionless, suspended from a spring with constant 100 newtons per metre. A vertical force $2 \sin \omega t$ acts on the mass beginning at time $t = 0$. Oscillations are subject to a damping force in newtons equal to twice the velocity in metres per second. Find the position of the mass as a function of time.

(b) What value of ω causes damped resonance? What is the amplitude of steady-state oscillations for damped resonance?

14. A mass M is suspended from a spring with constant k. Vertical motion is initiated by an external force $A \cos \omega t$ where A is a positive constant. During the subsequent motion a damping force acts on the mass with damping coefficient β.

(a) Show that the steady-state part of the solution is

$$x_p(t) = \frac{A(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2} \cos \omega t - \frac{A\omega\beta}{(k - M\omega^2)^2 + \beta^2\omega^2} \sin \omega t.$$

(b) Find the value of ω that gives damped resonance and the resulting amplitude of oscillations.

15. A battery of springs is placed between two sheets of wood, and the structure is placed on a level floor. Equivalent to the springs is a single spring with constant 1000 newtons per metre. A 20 kilogram mass is dropped onto the upper platform, hitting the platform with speed 2 metres per second, and remains attached to the platform thereafter.

(a) Find the position of the mass relative to where it strikes the platform as a function of time. Assume that air drag is 10 times the velocity of the mass.

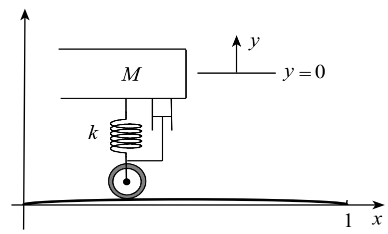
(b) What is the maximum displacement from where it strikes the platform experienced by the mass?

16. Use the techniques of this section to solve Exercise 42 in Section 3.2.

17. Use the techniques of this section to solve Exercise 43 in Section 3.2.

18. A mass of 500 grams is at equilibrium suspended from a spring with constant 250 newtons per metre. At time $t = 0$, the apparatus to which the top end of the spring is attached moves up and down sinusoidally according to $f(t) = 0.1 \sin 2t$ metres, where $f(t)$ is positive when the apparatus is above its starting position. If damping with coefficient $\beta = 10$ acts on the mass during its motion, find the position of the mass as a function of time. Describe the motion of the mass.

19. (a) In this exercise, we analyze vertical motion of a truck as it traverses a speed bump. The figure to the right shows part of the truck with mass $M = 500$ kilograms over a wheel. Between the wheel and the truck body is a spring with constant $k = 1000$ Newtons per metre. For the moment, we assume that the truck has no shocks to dampen oscillations. The



The truck hits the bump at time $t = 0$ traveling at constant speed $v = 18$ kilometres per hour. The equation of the bump (in metres) is $A \sin \pi x$, $0 \le x \le 1$. Show that the initial-value problem for displacement of the truck from its equilibrium position is

$$\frac{d^2y}{dt^2} + 2y = 2A \sin 5\pi t, \quad y(0) = 0, \quad y'(0) = 0.$$

For how long is this equation valid?

(b) Find $y(t)$.

20. A mass M, attached to the right end of a spring with constant k, rests on a horizontal table. The left end of the spring is attached to a wall. At time $t = 0$ the mass is pulled to the right a distance x_0 and given velocity v_0 to the right. If damping is ignored, but the coefficient of kinetic friction between table and mass is μ, find a formula for the time when the mass comes to an instantaneous stop for the first time.

21. Repeat Exercise 20 if the initial velocity is to the left.

22. A mass M, attached to the right end of a spring with constant k, rests on a horizontal table. The left end of the spring is attached to a moving support. At time $t = 0$ the mass is pulled to the right a distance x_0 and given velocity v_0 to the right. The motion of the support in the x-direction is described by $A \sin \omega t$, where $A > 0$ is a constant. If damping is ignored, find the position of the mass in the nonresonant case. What value of ω causes undamped resonance?

23. A cube 1 metre on each side and with density 1200 kilograms per cubic metre is placed with one of its faces in the surface of a body of water. When the cube is released from this position and sinks, it is acted upon by three forces, gravity, a buoyant force equal to the weight of water displaced by the submerged portion of the cube (Archimedes' principle), and a resistive force equal to twice the speed of the object. Find the depth of the bottom surface of the cube as a function of time from the instant the cube is released until it is completely submerged. Plot a graph of the function.

24. A cube 1 metre on each side and with density 500 kilograms per cubic metre is placed with one of its faces in the surface of a body of water. When the cube is released from this position and sinks, it is acted upon by three forces, gravity, a buoyant force equal to the weight of water displaced by the submerged portion of the cube (Archimedes' principle), and a resistive force equal to twice the speed of the object. Find the depth of the bottom surface of the cube as a function of time. Plot a graph of the function.

25. A cable hangs over a peg, 10 metres on one side and 15 metres on the other. Find the time for it to slide off the peg

(a) if friction at the peg is negligible.
(b) if friction at the peg is equal to the weight of 1 metre of cable.

5.4 LCR Circuits

If a resistor with resistance R, an inductor with inductance L, and a capacitor with capacitance C are connected in series with an electromotive force $E(t)$ (Figure 5.23), and the switch is closed, current flows in the circuit and charge builds up on the capacitor. If at any time t, Q is the charge on the capacitor and I is the current in the loop, then Kirchhoff's voltage law states that

$$L\frac{dI}{dt} + RI + \frac{Q}{C} - E(t) = 0, \tag{5.26}$$

where LdI/dt, RI, and Q/C represent voltage drops across the inductor, resistor, and capacitor, respectively. If we substitute $I = dQ/dt$, then

$$L\frac{d^2Q}{dt} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t), \tag{5.27}$$

a linear second-order differential equation for $Q(t)$. Alternatively, if we differentiate this equation, we obtain

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = E'(t), \tag{5.28}$$

a linear second-order differential equation for $I(t)$.

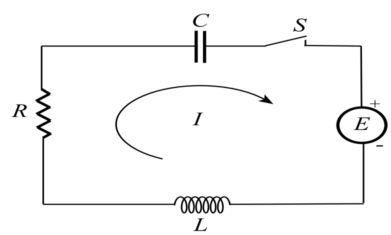


Figure 5.23

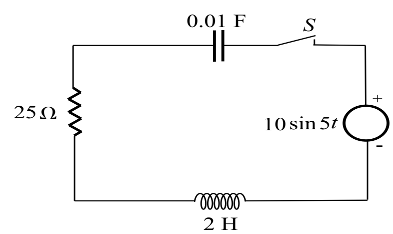


Figure 5.24

The similarity between differential equations 5.7 and 5.27 is unmistakable:

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t),$$
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Each of the coefficients M, β, and k for the mechanical system has its analogue L, R, and $1/C$ in the electrical system. This suggests that LCR circuits might be used to model complicated physical systems subject to vibrations, and conversely, that mass-spring systems might represent electrical systems. A full discussion of LCR circuits would parallel that in Sections 5.1–5.3. There should be no need to repeat all discussions in detail, and we therefore choose to do some representative problems.

Example 5.13 At time $t = 0$, a 25-Ω resistor, a 2-H inductor, and a 0.01-F capacitor are connected in series with a generator producing an alternating voltage of $10\sin 5t$, $t \geq 0$ (Figure 5.24). Find the charge on the capacitor and the current in the circuit if the capacitor is uncharged when the circuit is closed.

Solution The differential equation for the charge Q on the capacitor is

$$2\frac{d^2Q}{dt^2} + 25\frac{dQ}{dt} + 100Q = 10\sin 5t,$$

to which we add the initial conditions

$$Q(0) = 0, \qquad Q'(0) = I(0) = 0.$$

The auxiliary equation is $2m^2 + 25m + 100 = 0$ with solutions

$$m = \frac{-25 \pm \sqrt{625 - 800}}{4} = \frac{-25 \pm 5\sqrt{7}i}{4}.$$

Consequently, a general solution of the associated homogeneous equation is

$$Q_h(t) = e^{-25t/4}\left(C_1 \cos\frac{5\sqrt{7}t}{4} + C_2 \sin\frac{5\sqrt{7}t}{4}\right).$$

To find a particular solution of the nonhomogeneous equation by undetermined coefficients, we set $Q_p(t) = A\sin 5t + B\cos 5t$. Substitution into the differential equation gives

$$2(-25A\sin 5t - 25B\cos 5t) + 25(5A\cos 5t - 5B\sin 5t) + 100(A\sin 5t + B\cos 5t) = 10\sin 5t.$$

This equation requires A and B to satisfy

$$50A - 125B = 10, \qquad 125A + 50B = 0,$$

the solution of which is $A = 4/145$, $B = -10/145$. A particular solution of the differential equation is therefore

$$Q_p(t) = \frac{2}{145}(2\sin 5t - 5\cos 5t),$$

and a general solution is

$$Q(t) = e^{-25t/4}\left(C_1 \cos\frac{5\sqrt{7}t}{4} + C_2 \sin\frac{5\sqrt{7}t}{4}\right) + \frac{2}{145}(2\sin 5t - 5\cos 5t).$$

The initial conditions require

$$0 = Q(0) = C_1 - \frac{10}{145}, \qquad 0 = Q'(0) = -\frac{25}{4}C_1 + \frac{5\sqrt{7}}{4}C_2 + \frac{20}{145},$$

and these imply that $C_1 = 10/145$ and $C_2 = 34/(145\sqrt{7})$. Consequently, the charge on the capacitor, in coulombs, is

$$Q(t) = \frac{e^{-25t/4}}{145\sqrt{7}}\left(10\sqrt{7}\cos\frac{5\sqrt{7}t}{4} + 34\sin\frac{5\sqrt{7}t}{4}\right) + \frac{2}{145}(2\sin 5t - 5\cos 5t).$$

The current in the circuit, in amperes, is

$$\begin{aligned} I(t) = \frac{dQ}{dt} &= \left(-\frac{25}{4}\right)\frac{e^{-25t/4}}{145\sqrt{7}}\left(10\sqrt{7}\cos\frac{5\sqrt{7}t}{4} + 34\sin\frac{5\sqrt{7}t}{4}\right) \\ &\quad + \frac{e^{-25t/4}}{145\sqrt{7}}\left(-\frac{175}{2}\sin\frac{5\sqrt{7}t}{4} + \frac{85\sqrt{7}}{2}\cos\frac{5\sqrt{7}t}{4}\right) \end{aligned}$$

$$+\frac{2}{145}(10\cos 5t + 25\sin 5t)$$
$$= -\frac{e^{-25t/4}}{29\sqrt{7}}\left(4\sqrt{7}\cos\frac{5\sqrt{7}t}{4} + 60\sin\frac{5\sqrt{7}t}{4}\right)$$
$$+\frac{2}{29}(2\cos 5t + 5\sin 5t).$$

The solution $Q(t)$ contains two parts. The first two terms (containing the exponential $e^{-25t/4}$) are $Q_h(t)$ with constants C_1 and C_2 determined by the initial conditions; the last two terms are $Q_p(t)$. We point this out because the two parts exhibit completely different characteristics. For small t, both parts of $Q(t)$ contribute significantly, but for large t, the first two terms become negligible. In other words, after a long time, the charge $Q(t)$ on the capacitor is defined essentially by $Q_p(t)$. We call $Q_p(t)$ the **steady-state** part of the solution, and the two other terms in $Q(t)$ are called the **transient** part of the solution. Similarly, the first two terms in $I(t)$ are the transient part of the current and the last two terms are the steady-state current in the circuit.

Finally, note that the frequency of the steady-state part of either $Q(t)$ or $I(t)$ is exactly that of the forcing voltage $E(t)$.•

Example 5.14 The 25-Ω resistor, 2-H inductor, and 0.01-F capacitor of Example 5.13 are connected in series at time $t = 0$ with an electromotive force $E\sin\omega t$, where $E > 0$ is a constant. The charge on the capacitor is 0.01 coulombs and the current in the circuit is zero. Find the value of ω for which the amplitude of the steady-state part of the current is a maximum (the so-called damped resonance frequency). What is the maximum amplitude?

Solution The steady-state current is the particular solution of the differential equation as predicted by the method of undetermined coefficients,

$$I_p(t) = B\sin\omega t + D\cos\omega t.$$

Substitution into the differential equation

$$2\frac{d^2I}{dt^2} + 25\frac{dI}{dt} + 100I = \omega E\cos\omega t$$

gives

$$2[-\omega^2 B\sin\omega t - \omega^2 D\cos\omega t] + 25[\omega B\cos\omega t - \omega D\sin\omega t]$$
$$+ 100[B\sin\omega t + D\cos\omega t] = \omega E\cos\omega t.$$

When we equate coefficients of terms in $\sin\omega t$ and $\cos\omega t$, we obtain the equations

$$-2\omega^2 B - 25\omega D + 100B = 0,$$
$$-2\omega^2 D + 25\omega B + 100D = \omega E.$$

Solution of this system is

$$B = \frac{25\omega^2 E}{(100-2\omega^2)^2 + 625\omega^2}, \qquad D = \frac{\omega E(100-2\omega^2)}{(100-2\omega^2)^2 + 625\omega^2},$$

and the steady-state current is

$$I_p(t) = \frac{25\omega^2 E}{(100-2\omega^2)^2 + 625\omega^2}\sin\omega t + \frac{\omega E(100-2\omega^2)}{(100-2\omega^2)^2+625\omega^2}\cos\omega t.$$

The amplitude of this current is

$$A(\omega) = \frac{\sqrt{(25\omega^2 E)^2 + \omega^2 E^2(100-2\omega^2)^2}}{(100-2\omega^2)^2+625\omega^2} = \frac{\omega E}{\sqrt{(100-2\omega^2)^2+625\omega^2}}.$$

To determine the maximum amplitude, we set

$$0 = \frac{dA}{d\omega} = E\left\{\frac{1}{\sqrt{(100-2\omega^2)^2+625\omega^2}} - \frac{\omega[2(100-2\omega^2)(-4\omega)+1250\omega]}{2[(100-2\omega^2)^2+625\omega^2]^{3/2}}\right\}.$$

This simplifies to

$$\frac{4E(2500-\omega^4)}{[(100-2\omega^2)^2+625\omega^2]^{3/2}} = 0.$$

The positive solution is $\omega = 5\sqrt{2}$. Maximum amplitude occurs at this frequency, $A(5\sqrt{2}) = E/25$ amperes.●

EXERCISES 5.4

1. A 0.001-F capacitor and 2-H inductor are connected in series with a 20-V battery. If there is no charge on the capacitor before the battery is connected find the current in the circuit as a function of time.

2. At time $t = 0$, a 0.02-F capacitor, a 100-Ω resistor, and a 1-H inductor are connected in series. If the charge on the capacitor is initially 5 C, find its charge as a function of time.

3. A 5-H inductor and 20-Ω resistor are connected in series with a generating supplying an oscillating voltage of $10\sin 2t$, $t > 0$. What are the transient and steady-state currents in the circuit?

4. A time $t = 0$ an uncharged 0.1-F capacitor is connected in series with a 0.5-H inductor and a 3-Ω resistor. If the current in the circuit at this instant is 1 A, find the maximum charge that the capacitor stores.

5. A 25/9-H inductor, a 0.04-F capacitor, and a generator with voltage $15\cos 3t$ are connected in series at time $t = 0$. Find the current in the circuit as a function of time. Does resonance occur?

5.5 Beam Deflections

An important application of differential equations in structural engineering is to determine the shape of a horizontal beam when it is subjected to various loads. By analyzing internal forces and moments, it can be shown that the shape $y(x)$ of a uniform beam with constant cross section (Figure 5.25) is governed by the equation

$$\frac{d^4y}{dx^4} = \frac{F(x)}{EI} \tag{5.29}$$

where E is a constant called Young's modulus of elasticity (depending on the material of the beam), and I is also a constant (the moment of inertia of the cross section of the beam). Quantity $F(x)$ is the load placed on the beam; it is the vertical force per unit length in the x-direction, placed at position x, including the weight of the beam itself. For example, if a beam has mass 100 kilograms and length 10 metres (Figure 5.26), then the load due to its weight is a constant $F(x) = -9.81(100/10) = -98.1$ newtons per metre at every point of the beam.

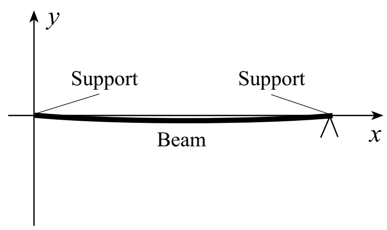


Figure 5.25

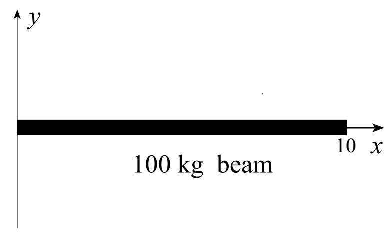


Figure 5.26

Suppose a block with mass 40 kilograms, uniform in cross section, and length 4 metres is centred on the beam in Figure 5.26 (see Figure 5.27). It adds an additional load of 9.81(10)=98.1 newtons per metre over the interval $3 < x < 7$. The total load is then a piecewise constant function,

$$F(x) = \begin{cases} -98.1, & 0 < x < 3 \\ -196.2, & 3 < x < 7 \\ -98.1, & 7 < x < 10. \end{cases} \tag{5.30}$$

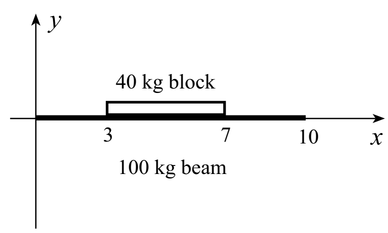


Figure 5.27

Heaviside unit step functions provide compact descriptions to functions like this. The fundamental unit step function is defined by

$$h(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases} \tag{5.31}$$

Its graph is shown in Figure 5.28; there is a discontinuity of magnitude unity at $x = 0$. (Some authors replace $x \geq 0$ in this definition with $x > 0$ so that the function is undefined at $x = 0$. The rest of this section can be developed with either convention with minor adjustments in results.)

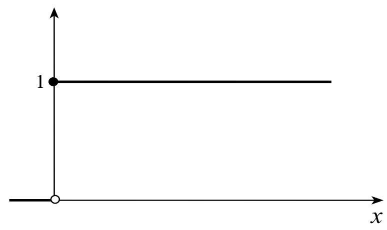


Figure 5.28

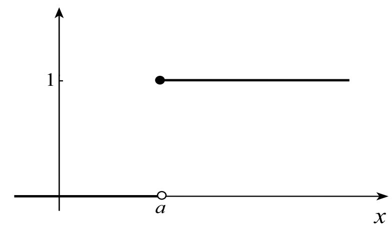


Figure 5.29

When the discontinuity occurs at $x = a$, the function is denoted by

$$h(x-a) = \begin{cases} 0, & x < a \\ 1, & x > a. \end{cases} \tag{5.32}$$

Its graph is shown in Figure 5.29.

Heaviside unit step functions provide compact descriptions to functions with finite jump discontinuities. One of the most important is shown in Figure 5.30. It is called a **pulse** function. It can be expressed algebraically in the form $h(x-a) - h(x-b)$, except at $x = a$ and $x = b$. In the event that the height of the nonzero portion is c rather than unity (Figure 5.31), we obtain $c[h(x-a) - h(x-b)]$, again except at $x = a$ and $x = b$.

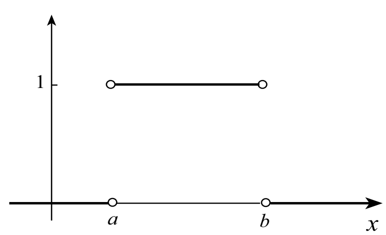


Figure 5.30

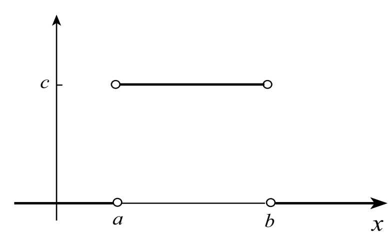


Figure 5.31

Pulse functions can be combined algebraically to produce step functions. The function in Figure 5.32 is the sum of two pulse functions,

$$4[h(x) - h(x-3)] + 2[h(x-3) - h(x-6)] = 4h(x) - 2h(x-3) - 2h(x-6),$$

except at $x = 0, 3,$ and 6. The function in Figure 5.33 is the sum of three pulses,

$$\begin{aligned} 3[h(x-a) - h(x-b)] + 4[h(x-b) - h(x-c)] + h(x-c) \\ = 3h(x-a) + h(x-b) - 3h(x-c), \end{aligned}$$

except at $x = a, b,$ and c. In future representations of piecewise defined functions in terms of Heaviside functions, we will drop the exceptions.

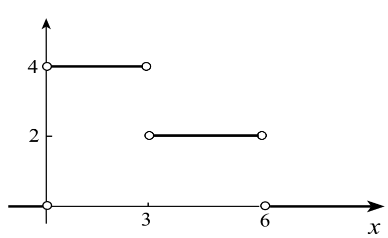


Figure 5.32

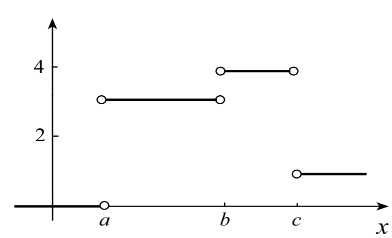


Figure 5.33

The function in equation 5.30 can be described as follows

$$F(x) = -98.1[h(x) - h(x-3)] - 196.2[h(x-3) - h(x-7)] - 98.1[h(x-7) - h(x-10)]$$
$$= -98.1h(x) - 98.1h(x-3) + 98.1h(x-7) + 98.1h(x-10).$$

Since the beam is confined to the interval $0 \le x \le 10$, $h(x)$ is always equal to unity, and the function $h(x-10)$ is redundant,

$$F(x) = -98.1 - 98.1h(x-3) + 98.1h(x-7).$$

Accompanying differential equation 5.29 will be four boundary conditions defining the type of support (if any) at each end of the beam. Three types of supports are common. We discuss them at the left end of the beam, but they also occur at the right end.

1. **Simple Support**

The end of a beam is simply-supported when it cannot move vertically, but it is free to rotate. Visualize that a horizontal pin perpendicular to the xy-plane passes through a hole in the end of the beam at $x = 0$ (Figure 5.34). The pin is fixed, but the end of the beam can rotate on the pin. In this case, $y(x)$ must satisfy the **boundary conditions**

$$y(0) = y''(0) = 0. \tag{5.33a}$$

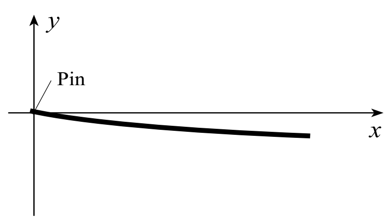


Figure 5.34

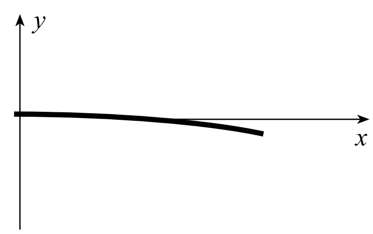


Figure 5.35

2. **Built-in End**

If the end $x = 0$ of the beam is permanently fixed in a horizontal position, embedded horizontally in concrete say, (Figure 5.35), $y(x)$ satisfies

$$y(0) = y'(0) = 0. \tag{5.33b}$$

3. **Free Support (or Cantilevered)**

If the end $x = 0$ of the beam is not supported, perhaps like the end of a diving board, (Figure 5.36), $y(x)$ satisfies

$$y''(0) = y'''(0) = 0. \tag{5.33c}$$

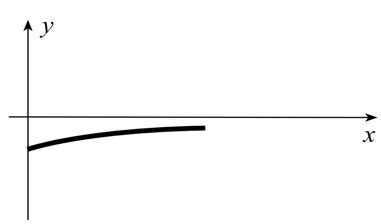


Figure 5.36

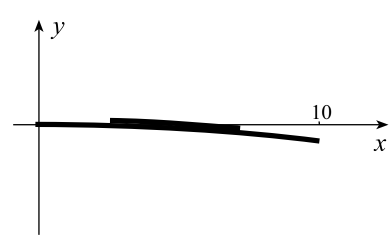


Figure 5.37

When two boundary conditions at each end of a beam accompany differential equation 5.29, we have what is called a **boundary-value problem**. For example, if the end $x = 0$ of the beam in Figure 5.27 is horizontally built-in, and the right end is free, just like a diving board (Figure 5.37), the boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}[-98.1 - 98.1\,h(x-3) + 98.1\,h(x-7)],$$
$$y(0) = y'(0) = 0, \quad y''(10) = y'''(10) = 0.$$

To solve this problem with the techniques of Chapter 4 is particularly uninviting. We would solve the differential equation on the intervals $0 < x < 3$, $3 < x < 7$, $7 < x < 10$, and match $y(x)$, $y'(x)$, $y''(x)$, and $y'''(x)$ at $x = 3$ and $x = 7$. An alternative is to use the following antiderivatives of the Heaviside function:

$$\int h(x-a)\,dx = (x-a)h(x-a) + C, \tag{5.34a}$$

$$\int (x-a)h(x-a)\,dx = \frac{1}{2}(x-a)^2h(x-a) + C, \tag{5.34b}$$

$$\int (x-a)^2h(x-a)\,dx = \frac{1}{3}(x-a)^3h(x-a) + C, \tag{5.34c}$$

$$\int (x-a)^3h(x-a)\,dx = \frac{1}{4}(x-a)^4h(x-a) + C. \tag{5.34d}$$

Even better is the Laplace transform to be discussed in Chapter 6. Not only does it handle discontinuous loads very efficiently, it also allows us to consider concentrated loads, something that we cannot handle with the techniques of Chapter 4, nor with the above formulas. In the meantime, we demonstrate the use of formulas 5.34.

Example 5.15 A uniform diving board with fixed end at $x = 0$ has length L and mass m. Find its deflections.

Solution The boundary-value problem for deflections of the board is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left(\frac{-9.81m}{L}\right), \qquad y(0) = y'(0) = 0, \quad y''(L) = y'''(L) = 0.$$

Four antidifferentiations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL}.$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI}, \quad 0 = 6C_4 - \frac{9.81m}{EI}.$$

These imply that $C_3 = -\dfrac{2.4525mL}{EI}$ and $C_4 = \dfrac{1.635m}{EI}$, and deflections of the board are

$$y(x) = \frac{1}{EI}\left(-2.4525mLx^2 + 1.635mx^3 - \frac{9.81mx^4}{24L}\right). \bullet$$

Example 5.16 A uniform beam of length L has mass m. An additional mass M is distributed uniformly over the right half of the beam. If the left end of the beam is fixed horizontally and the right end is simply-supported, find deflections of the beam.

Solution The load on the beam can be written in terms of Heaviside functions as

$$F(x) = -\frac{9.81m}{L} - \frac{9.81M}{L/2}[h(x - L/2) - h(x - L)].$$

Because $h(x - L) = 0$ for $0 < x < L$, we can simplify this to

$$F(x) = -\frac{9.81m}{L} - \frac{19.62M}{L}h(x - L/2).$$

With this, the boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left[\frac{-9.81m}{L} - \frac{19.62M}{L}h(x - L/2)\right], \qquad y(0) = y'(0) = 0, \quad y(L) = y''(L) = 0.$$

Four integrations of this equation using equations 5.34 give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL} - \frac{19.62M}{24EIL}(x - L/2)^4h(x - L/2).$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = C_1 + C_2L + C_3L^2 + C_4L^3 - \frac{9.81mL^3}{24EI} - \frac{9.81M}{12EIL}(L - L/2)^4,$$
$$0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI} - \frac{9.81M}{EIL}(L - L/2)^2.$$

These imply that

$$C_3 = -\frac{9.81mL}{16EI} - \frac{21(9.81)ML}{384EI}, \qquad C_4 = \frac{5(9.81)m}{48EI} + \frac{23(9.81)M}{384EI}.$$

Deflections of the beam are

$$\begin{aligned} y(x) &= \left[-\frac{9.81mL}{16EI} - \frac{21(9.81)ML}{384EI}\right]x^2 + \left[\frac{5(9.81)m}{48EI} + \frac{23(9.81)M}{384EI}\right]x^3 \\ &\quad - \frac{9.81mx^4}{24EIL} - \frac{19.62M}{24EIL}(x - L/2)^4h(x - L/2) \\ &= \frac{9.81}{384EIL}\left[-(24m + 21M)L^2x^2 + (40m + 23M)Lx^3 \right. \\ &\qquad \left. - 16mx^4 - 32M(x - L/2)^4h(x - L/2)\right].\bullet \end{aligned}$$

With the exception of Exercises 31 and 32 in Section 4.5, this is our first encounter with linear differential equations that have discontinuous nonhomogeneities. We noted in Section 4.1, that when nonhomogeneity $F(x)$ in differential equation 4.1 is continuous (as are all the coefficient functions), then the solution of the differential equation has continuous derivatives up to and including order n. If we examine the solution of Example 5.16, we find that its first three derivatives are continuous, but not the fourth. In general, a finite discontinuity in the nonhomogeneity of an n^{th}-order differential equation is reflected in a finite discontinuity in the n^{th} derivative of the solution.

EXERCISES 5.5

1. Find deflections of a uniform beam with mass m and length L when both ends are simply supported.

2. Repeat Exercise 1 if both ends of the beam are fixed horizontally.

3. Repeat Exercise 1 if the left end of the beam is fixed horizontally and the right end is simply supported.

4. Repeat Example 5.16 if the extra load is on the left half of the beam.

5. Repeat Example 5.16 if the extra load is over the middle half of the beam.

6. Repeat Example 5.15 if an additional mass M is distributed uniformly over the right half of the board. What is the deflection of the right end of the board?

7. Repeat Example 5.15 if an additional mass M is distributed uniformly over the left half of the board. How does the deflection of the right end of the board compare to that in Exercise 6?

8. Repeat Example 5.15 if an additional mass M is distributed uniformly over the middle half of the board. How does the deflection of the right end of the board compare to that in Exercises 6 and 7?