Some Results in Finite Graph Ramsey Theory

David S. Gunderson
University of Calgary

Vojtěch Rödl
Emory University, Atlanta

Norbert W. Sauer*
University of Calgary

Abstract

For given finite graphs G and H, when can we assert the existence of a Ramsey graph F with $F \longrightarrow (G)_r^H$? If there exists an ordering $\leq$ of G so that all H-subgraphs of $(G, \leq)$ are order-isomorphic, then as an easy consequence of the Ramsey theorem for ordered hypergraphs [e.g. Nešetřil, Rödl, 77,83], such an F exists. It is natural to conjecture that this is not only a sufficient condition for the existence of such an F, but also necessary. We show the situation to be more complicated than this and present a small counterexample to this conjecture. An infinite family of such counterexamples is then given. We also characterize those triples G, H and r for which a Ramsey F exists.
KEYWORDS: Chromatic number, graph Ramsey theory, ordered hypergraph, probabilistic method.

1 Notation

We sometimes use ordinal representation $n = \{0, 1, 2, \ldots, n-1\}$ for the non-negative integers, so the notation $0 \leq m < n$ may be replaced by $m \in n$ whenever clear. For a set S and a given $n \in \omega$ we define $[S]^n = \{T \subseteq S : |T| = n\}$ to be the set of all subsets of S of size n. Similarly we define $[S]^{\leq n}$.

We define a hypergraph $(X, \mathcal{E})$ on the vertex set X by a collection of maps $\alpha_0, \ldots, \alpha_{m-1}$, $\alpha_i : [X]^{k_i} \to n_i$ for integers $n_i \in \omega$ and k_i satisfying $1 \leq k_i \leq |X|$. The edge set $\mathcal{E}$ can be interpreted as the collection of all subsets $Y \subset X$ such that $\alpha_i(Y) > 0$ for some $i \in m$. The image $\alpha_i(Y) \in n_i$ can be interpreted as the 'type' or 'multiplicity' of the edge Y. For example, if $n_i = 2 = \{0, 1\}$ for each i, then there are only edges of one type. A *k-uniform hypergraph* (with no loops or multiple edges) has $m = 1$, $k_0 = k$ and $n = 2$. For ordinary graphs (with no loops or multiple edges) there is only one map $\alpha : [X]^2 \to 2$.

Unless otherwise specified, graphs are meant to be finite, loopless, and without multiple edges. We use $G = (V(G), E(G))$ to mean G is a hypergraph on the vertex set $V(G)$ with edges $E(G) \subseteq 2^{[V(G)]}$. If H is a *weak subhypergraph* of G, i.e. $V(H) \subset V(G)$ and

*This research has been supported in part by NSERC grant 63-1325.

$E(H) \subseteq 2^{[V(H)]} \cap E(G)$ then we write $H \subseteq G$. If $H \subseteq G$ and $E(H) = 2^{[V(H)]} \cap E(G)$ then we say H is an *induced* subhypergraph of G, denoted by $H \preceq G$. Letting $\cong$ denote graph isomorphism, we define

$$\binom{G}{H} = \{H' \preceq G : H' \cong H\}.$$

An *ordered hypergraph* $(G, \leq)$ is a hypergraph G together with a total order $\leq$ on $V(G)$. Two ordered hypergraphs are isomorphic just in case there is an order preserving graph isomorphism between them. Definitions analagous to those given above hold for ordered hypergraphs as well. For a hypergraph H we let

$$ORD(H) = \{(H, \leq_0), (H, \leq_1), \ldots, (H, \leq_{k-1})\},$$

be the set of (distinct) isomorphism types of orderings of H. It is often convenient to abuse the notation and deliberately confuse an isomorphism type with a hypergraph of that given type.

For this discussion we introduce some new notation. For a given (unordered) hypergraph H and an ordered hypergraph $(G, \leq^*)$ we define

$$DO(H, G, \leq^*) = \{(H, \leq_i) \in ORD(H) : \binom{G, \leq^*}{H, \leq_i} \neq \emptyset\}.$$

Set $mdo(H, G) = \min\{|DO(H, G, \leq^j)| : (G, \leq^j) \in ORD(G)\}$, denoting the smallest number of orderings of H in any one ordered G . For example, if an ordinary graph H is complete, then $mdo(H, G) \leq 1$ for any choice of G. The number $mdo(H, G)$ will be of particular interest throughout this paper.

For hypergraphs F, G and H, and a fixed $r \in \omega$, we use the standard Ramsey arrow notation $F \longrightarrow (G)^H_r$ to mean that for any coloring $\Delta : \binom{F}{H} \longrightarrow n$, there exists $G' \in \binom{F}{G}$ so that Δ is constant on $\binom{G'}{H}$. We use the analagous notation for ordered graphs. We introduce further the special notation

$$\mathcal{R}[(G)^H_r] = \{F : F \longrightarrow (G)^H_r\},$$

the *Ramsey class* for G in coloring of H's with r colors. Observe that for these Ramsey type statements to be non-trivial we usually only consider pairs G, H so that $mdo(H, G) \geq 1$.

In ordinary graphs, we use P_n to refer to a path of length n on $n+1$ vertices. A cycle on k vertices is denoted by C_k and a complete graph on n vertices by K_n.

2 Introduction

A familiar Ramsey statement is: $6 \longrightarrow (3, 3)$. This says that if we color the pairs of a six element set with two colors then we are guaranteed the existence of a three element subset, all of whose two element subsets are colored the same. Translated into the language of graph theory, this statement reads: $K_6 \longrightarrow (K_3)^{K_2}_2$.

The finite Ramsey's theorem [16] can be stated as follows:

Theorem 2.1 *For any m,k,$r \in \omega$, there exists an $n \in \omega$ so that $K_n \longrightarrow (K_m)_r^{K_k}$, i.e., $\mathcal{R}[(K_m)_r^{K_k}] \neq \emptyset$.*

In general, if we are given two graphs G and H and a number $r \in \omega$, it is quite difficult to ascertain whether or not there is a graph $F \in \mathcal{R}[(G)_r^H]$. One of the earlier successes [8] is the following

Theorem 2.2 *For any ordinary graph G, $\mathcal{R}[(G)_2^{K_1}] \neq \emptyset$.*

Proof: Let the graph G be given and define the lexicographic product $F = H \otimes H$ on $V(F) = V(G) \times V(G)$ by

$$((u_0, v_0), (u_1, v_1)) \in E(F) \text{ iff } \begin{cases} (u_0, v_0) \in E(G) & \textbf{or} \\ u_0 = u_1 \text{ and } (v_0, v_1) \in E(G) \end{cases}$$

It can be verified that F satisfies $F \longrightarrow (G)_2^{K_1}$ (since if there is no monochromatic copy of G in any of the 'coordinates' of F, then there is certainly one straddling the coordinates).□

Theorem 2.3 ([2],[6] and [17]) *For any ordinary graph G, $\mathcal{R}[(G)_2^{K_2}] \neq \emptyset$.*

A proof of this theorem is already quite difficult, and further questions of this nature may be increasingly stubborn. (More recently, a very readable proof of Theorem 2.3 using a partite construction is given in [11].) Could it be that for *every* triple G, H and r we have a Ramsey $F \in \mathcal{R}[(G)_r^H]$? This can be answered in the negative by the following well known example (e.g. [14], p.192):

Example 2.4 $\mathcal{R}[(C_4)_2^{P_2}] = \emptyset$.

Proof: Let $\leq^*$ be any total order of $V(C_4)$. Then it is easy to see that $(C_4, \leq^*)$ contains at least two distinct orderings of P_2, namely one with the middle vertex highest in the order, and one with the middle vertex lowest in the order. (These two 'middle' vertices correspond to the two vertices on the 'ends' of the order in $(C_4, \leq^*)$.)

Now fix any (ordinary) graph F. Impose an arbitrary order $\leq$ on $V(F)$. We will produce a coloring of $\binom{F}{P_2}$ which ensures that every copy of C_4 in F is multicolored. Simply color the copies of P_2 according to their orientation; if one is 'pointed' upwards, color it red and if one is pointed downwards, color it blue. We can color the other ordered P_2's arbitrarily. Now since each ordered C_4 contains one of each kind of P_2 it recieves two colors. □

It is not difficult to see that $mdo(P_2, C_4) = 2$. In subsequent sections we rely heavily on this idea of ordering graphs so that we can find particular colorings. Pairs like C_4 and P_2 are not anomalous; there are 'many' such cases, as given in the following theorem [9].

We call a graph *trivial* if it is either complete or empty.

Theorem 2.5 *For every non-trivial graph H there is a graph G so that $\mathcal{R}[(G)_2^H] = \emptyset$.*

The proof of this theorem uses an idea similar to the one used for Example 2.4, that is, the idea of coloring ordered graphs.

3 The Question

For a fixed $r \in \omega$ and given graphs G and H, how can we tell if $\mathcal{R}[(G)^H_r] \neq \emptyset$? Oddly enough, this is completely answered in the case of ordered graphs. This might seem counter-intuitive since ordered graphs are rigid and so any Ramsey structure would have to be larger, in some sense, than in the unordered case so as to contain the necessary richness of substructures required. Nevertheless ...

Theorem 3.1 *Given $r \in \omega$ and ordered hypergraphs $(G, \leq)$ and $(H, \leq)$,*

$$\mathcal{R}[(G, \leq)^{(H,\leq)}_r] \neq \emptyset.$$

This theorem is due to Nešetřil and Rödl [10],[12],[13] and independently, Abramson and Harrington [1]. We omit the proof. An immediate application of this powerful theorem is the following:

Corollary 3.2 *Fix $r \in \omega$. If H and G are (unordered) hypergraphs with $mdo(H, G) = 1$ then $\mathcal{R}[(G)^H_r] \neq \emptyset$.*

Proof: Let $mdo(H, G) = 1$ and fix an ordering $\leq$ of G so that every induced H-subgraph of G is $\leq$-order-isomorphic to say $(H, \leq)$. Apply Theorem 3.1 to obtain $(F, \leq) \in \mathcal{R}[(G, \leq)^{(H,\leq)}_r]$. We claim the unordered F also satisfies $F \longrightarrow (G)^H_r$.

Fix a coloring $\Delta : \binom{F}{H} \longrightarrow r$ and order F according to $\leq$. Then Δ induces a coloring $\Delta^* : \binom{F,\leq}{H,\leq} \longrightarrow r$ and so there exists a $(G', \leq) \in \binom{F,\leq}{G,\leq}$ so that Δ^* is constant on $\binom{G',\leq}{H,\leq}$. But since $|DO(H, G, \leq)| = 1$, Δ^* assigns a color to every copy of H in G'. Hence $\binom{G'}{H}$ is monochromatic with respect to Δ^* and so also with respect to Δ. $\square$

Now another otherwise difficult result proven by Deuber [3] and Nešetřil and Rödl [9], which is an extension of Theorems 2.2 and 2.3 of the previous section, is simply an obvious

Corollary 3.3 *For any ordinary graph G and fixed $r, n \in \omega$ $\mathcal{R}[(G)^{K_n}_r] \neq \emptyset$.*

One might hope to find some necessary conditions on G, H and r so that $\mathcal{R}[(G)^H_r] \neq \emptyset$, however at least one straightforward restriction must be respected.

Lemma 3.4 *Fix hypergraphs G and H where $|ORD(H)| = r$. If $mdo(H, G) \geq 2$ then $\mathcal{R}[(G)^H_r] = \emptyset$.*

Proof: In hope of a contradiction, suppose F is so that $F \longrightarrow (G)^H_r$ and impose an arbitrary ordering $\leq$ on $V(F)$. Now define an r-coloring $\Delta : \binom{F}{H} \longrightarrow r$ by $\Delta(H') = i$ if $(H', \leq) \cong (H, \leq_i) \in ORD(H)$. Since $mdo(H, G) \geq 2$, every copy of G in F is two-colored. $\square$

4 Counterexample

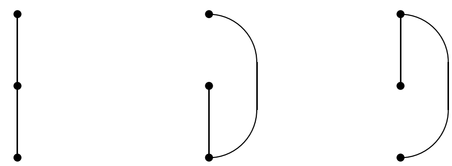
In a corollary ([15] p.54) Prömel and Voigt state that for all hypergraphs G and H, $mdo(H,G) = 1$ if and only if $\mathcal{R}[(G)_2^H] \neq \emptyset$.

After trying to prove this 'corollary' we discovered the following counterexample.

Lemma 4.1 *There exist graphs G and H so that $mdo(H,G) = 2$ but $\mathcal{R}[(G)_2^H] = \emptyset$.*

Proof: Let T and $P = P_2$ be the graphs given in Figure 1.
Label $ORD(P) = \{(P, \leq_0), (P, \leq_1)(P, \leq_2)\}$ as in Figure 2.

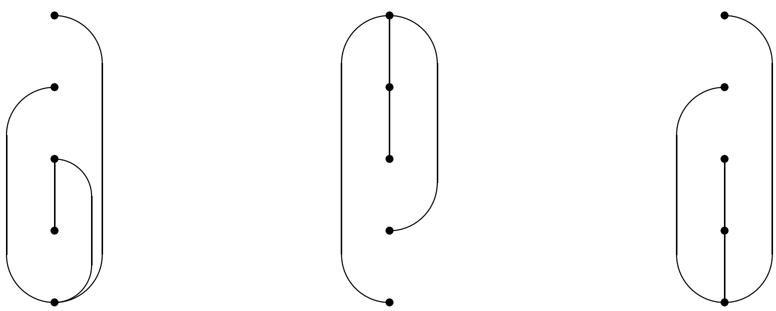
Figure 1: The graphs used for counterexample.



$(P, \leq_0)$ $(P, \leq_1)$ $(P, \leq_2)$

Figure 2: The orderings of P

It is easy to verify that $mdo(P,T) = 2$. Fix three orderings of T as shown in Figure 3.



$(T, \leq^0)$ $(T, \leq^1)$ $(T, \leq^2)$

Figure 3: Three orderings of T.

We point out that for each $i \in 3$, $(T, \leq^i)$ contains as induced P-subgraphs exactly $\{(P, \leq_j) : j \in 3, j \neq i\}$. Let $(B, \leq)$ be the ordered (disjoint) sum of $(T, \leq^0)$, $(T, \leq^1)$, and $(T, \leq^2)$, taken in a fixed, but arbitrary order.

Using Theorem 3.1 successively choose ordered graphs $(C, \leq)$, $(D, \leq)$ and $(F, \leq)$ so that

$$(C, \leq) \longrightarrow (B, \leq)_2^{(P, \leq_0)}, \tag{1}$$

$$(D, \leq) \longrightarrow (C, \leq)_2^{(P, \leq_1)}, \tag{2}$$

and

$$(F, \leq) \longrightarrow (D, \leq)_2^{(P, \leq_2)}. \tag{3}$$

We claim that F, the unordered version of $(F, \leq)$, actually satisfies $F \longrightarrow (T)_2^P$. Fix a 2-coloring $\Delta : \binom{F}{P} \longrightarrow 2$. By (3), there is $(D', \leq) \in \binom{F, \leq}{D, \leq}$ so that Δ is constant on $\binom{D', \leq}{P, \leq_3}$, say $\Delta[\binom{D', \leq}{P, \leq_2}] = s_2 \in 2$. Now by (2) there also exists $(C', \leq) \in \binom{D', \leq}{C, \leq}$ so that Δ is constant on $\binom{C', \leq}{P, \leq_1}$, say $\Delta[\binom{C', \leq}{P, \leq_1}] = s_1$ (while of course Δ is still constant on $\binom{C', \leq}{P, \leq_2}$). Similarly, by (1) we choose $(B', \leq) \in \binom{C', \leq}{B, \leq}$ with $\Delta[\binom{B', \leq}{P, \leq_0}] = s_0 \in 2$ while still being constant on $\binom{B', \leq}{P, \leq_2}$ and $\binom{B', \leq}{P, \leq_1}$. So in $(B', \leq)$ all copies of P are colored with two colors, only depending on their orientation. Since $\{s_0, s_1, s_2\} \subset 2$, at least two of s_0, s_1, s_2 agree. If, say, $s_0 = s_1$ then the $(T, \leq^2)$ part of $(B', \leq)$ has all its P-subgraphs colored the same. In any case, at least one monochromatic copy of T will exist as an induced subgraph of F. $\square$

In general, the idea is easy to apply if we can find an H with $|ORD(H)| = 3$, and G so that $mdo(H, G) \geq 2$ and yet there are 3 orderings of G witnessing the fact, each containing a different pair of (distinct) elements from $ORD(H)$ as induced subgraphs. This recipe can be generalized to reveal the essence of the method, as we see in the next section.

5 A characterization

Let $K = (X, \mathcal{K})$ be a hypergraph and recall that the *chromatic number*, $\chi(K)$, of K is the least integer n so that there is an n-coloring of the vertex set X yielding no monochromatic edge $E \in \mathcal{K}$. For a given pair of hypergraphs G and H, let us define a new hypergraph $K_{H,G}$ on the vertex set $ORD(H)$ with edge set $E(K_{H,G}) = \{DO(H, G, \leq^j) : (G, \leq^j) \in ORD(G)\}$. Since for each edge there corresponds an ordering of G we may, by abuse of notation, refer to the edges as orderings of G, i.e., we could say $E(K_{H,G}) = ORD(G)$, and a vertex $(H, \leq_i)$ is contained by an edge $(G, \leq^j)$ if and only if $(H, \leq_i) \preceq (G, \leq^j)$. We now give a characterization of those triples H, G and r for which there exists a Ramsey graph.

Theorem 5.1 *Given hypergraphs G and H, $\mathcal{R}[(G)_r^H] \neq \emptyset$ if and only if $\chi(K_{H,G}) > r$.*

The proof in one direction is based on the construction given in the proof of the counterexample and the other direction is by simple contradiction. It might be helpful to keep in mind that if $\chi(K_{H,G}) > r$ this would mean that for every r-coloring $\chi : ORD(H) \longrightarrow r$ there exists an order $\leq^*$ of G so that $DO(H, G, \leq^*)$ is monochromatic. This fact will be used to show that the graph we construct in the first part of the proof is indeed in $\mathcal{R}[(G)_r^H]$.

Throughout the proof we fix $r \in \omega$, hypergraphs G, H and $K = K_{G,H}$.
Proof:($\Leftarrow$) Assume $\chi(K) > r$. Enumerate

$$ORD(H) = \{(H, \leq_0), (H, \leq_1), \dots, (H, \leq_{t-1})\}$$

and

$$ORD(G) = \{(G, \leq^0), (G, \leq^1), \dots, (G, \leq^{s-1})\}.$$

Construct the graph $(B, \leq) = \dot{\bigcup}_{j \in s}(G, \leq^j)$, the (disjoint) ordered sum of the orderings of G. (It is not necessary that all the vertices of one ordering of G be entirely below all vertices of another, – though it helps to imagine it this way – only that the order of each is preserved and they remain disjoint, but yet form a new ordered graph.) By Theorem 3.1 choose $(B_0, \leq)$ satisfying $(B_0, \leq) \longrightarrow (B, \leq)_r^{(H,\leq_0)}$ and for $i = 1, \dots, t-1$ choose (again by Theorem 3.1) successively $(B_i, \leq)$ so that $(B_i, \leq) \longrightarrow (B_{i-1}, \leq)_r^{(H,\leq_i)}$.

We claim that B_{t-1}, the unordered version of $(B_{t-1}, \leq)$, satisfies $B_{t-1} \longrightarrow (G)_r^H$. Fix a coloring $\Delta : \binom{B_{t-1}}{H} \longrightarrow r$. As in the proof of Lemma 4.1, construction guarantees the existence of $(B', \leq) \in \binom{B_{t-1},\leq}{B,\leq}$ so that for any fixed i, all the induced $(H, \leq_i)$-subgraphs of $(B', \leq)$ are monochromatic. This coloring of ordered H's in $(B', \leq)$ induces a r-coloring χ of the vertices of $K_{H,G}$ and hence (by the remark preceding the proof) there exists a $(G, \leq^j)$ in the edge set of $K_{H,G}$ which is monochromatic (since $\chi(K_{H,G}) > r$) with respect to χ. Thus, there exists $G^* \in \binom{B_{t-1}}{G}$ monochromatic with respect to Δ.

($\Rightarrow$) Assume $\chi(K_{H,G}) \leq r$. So choose a coloring $\chi : ORD(H) \longrightarrow r$ so that each element in $ORD(G)$ is multi-colored. Choose any F and impose an arbitrary (but fixed) ordering $\leq^*$ on $V(F)$.. This naturally imposes an order on each $H' \in \binom{F}{H}$, so color $\binom{F}{H}$ according to χ. That is, define $\Delta : \binom{F}{H} \longrightarrow r$ by $\Delta(H') = \chi((H', \leq^*))$ for each $H' \in \binom{F}{H}$, where $(H', \leq^*) \in ORD(H)$ is the $\leq^*$-ordered H-subgraph. Then since each element in $ORD(G)$ is multi-colored with respect to χ, so also is each $G' \in \binom{F}{G}$ with respect to Δ. $\Box$

Theorem 5.1 now yields the following characterization which was also suggested to the authors by Xuding Zhu (oral communication).

Corollary 5.2 *For given hypergraphs G and H, $mdo(H, G) = 1$ if and only if for every $r \in \omega$ $\mathcal{R}[(G)_r^H] \neq \emptyset$.*

Proof: One direction is simply Corollary 3.2, so assume that for some fixed G and H and every $r \in \omega$, $\mathcal{R}[(G)_r^H] \neq \emptyset$. Then by Theorem 5.1 the chromatic number of the associated $K_{H,G}$ is infinite. Since $K_{H,G}$ is finite, this means that there is a hyperedge consisting of only one point. A single vertex edge would correspond to an ordering of G witnessing $mdo(H, G) = 1$. $\Box$

Additionally, one can now derive the following corollary of Theorem 5.1 by only examining a particular $K_{H,G}$ with known chromatic number. This gives sufficient conditions (which can be tested directly) on pairs of hypergraphs G and H for which $\mathcal{R}[(G)_r^H] \neq \emptyset$ holds.

Corollary 5.3 *Let G and H be hypergraphs with $mdo(H, G) = l \leq k = |ORD(H)|$ and fix $r \in \omega$. If there exists an s, $(l \leq s < k)$, so that both $k \geq rs - 1$* and *for each $J \subseteq ORD(H)$ with $|J| = s$ there exists $(G, \leq^j) \in ORD(G)$ so that $DO(H, G, \leq^j) = J$, then $\mathcal{R}[(G)_r^H] \neq \emptyset$.*

6 A special hypergraph

We will, in the next section, give an infinite family of pairs of graphs H and G so that $mdo(H,G) \geq 2$ however $\mathcal{R}[(G)^H_2]$ is non-empty. To do this we must first find a (very large) hypergraph with certain properties.

Let us recall the following definition of a hypergraph with no short cycles (cf. [5] p.94). For the r-uniform hypergraph $E = (X, \mathcal{E})$ has $girth(E) > l$ if for every sequence of distinct edges $f_0, f_1, \ldots, f_{j_0-1} \in \mathcal{E}$ with $j_0 \leq l$

$$|\bigcup_{j \in j_0} f_j| \ \geq j_0(r-1)+1 \tag{4}$$

holds. If (4) fails to be true, then a cycle of length $\leq j_0$ exists among $f_0, f_1, \ldots, f_{j_0-1}$.

For a hypergraph on a vertex set X partitioned by $X = X_0 \dot\cup X_1 \dot\cup \ldots \dot\cup X_{n-1}$, we use the notation $((X_i)_{i\in n}, \mathcal{E})$, commonly used for n-partite graphs. We denote by $\lceil x \rceil$ the least integer $z \geq x$ and $\lfloor x \rfloor$ is the greatest integer $y \leq x$.

Theorem 6.1 *For a given integers $n \geq 1$ and $l \geq 2$ there exists a $2n$-uniform hypergraph $E = ((X)_{i\in n}, \mathcal{E})$ with $|X_0| = |X_1| = \ldots = |X_{n-1}|$, which enjoys the following properties:*

1. *For each edge $e \in \mathcal{E}$, $|e \cap X_i| = 2$ for all $i \in n$.*
2. *Girth$(E) > l$*
3. *For each choice of $X'_0, X'_1, \ldots, X'_{n-1}$ with $X'_i \subset X_i$ and $|X'_i| \geq \frac{1}{n}|X_i|$ for all $i \in n$, $\mathcal{E} \cap [\bigcup_{i\in n} X'_i]^{2n} \neq \emptyset$, i.e., there exists an edge in the hypergraph induced by $\bigcup_{i\in n} X'_i$.*

For this proof we use the *probabilistic method*, due to P. Erdős [4], in a manner similar to the that used in [5]. We use the notation $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$ for functions f and g. Note that if $f(n) = o(n)$ then so also $f(n) = o(kn)$ for fixed $k > 0$.
Proof: The case $n = 1$ is trivial so consider only when $n \geq 2$ (while of course $l \geq 2$ as well). We use elementary asymptotic formulae (see [7]) throughout; it will be tacitly assumed that numbers used are large enough so these formulae hold. Let $N = N(n,l)$ be a large positive integer. Consider n pairwise disjoint sets $X_0, X_1, \ldots, X_{n-1}$ of cardinality N.

Let $\mathcal{U}$ be the set of all $2n$-element sets f with the property $|f \cap X_i| = 2$ for every $i \in n$. Fix $\epsilon = \frac{1}{2l}$ and set $M = \lceil N/n \rceil$. Let $\boldsymbol{\mathcal{E}}$ be random subset of $\mathcal{U}$, where for $f \in \mathcal{U}$

$$\text{Prob}[f \in \boldsymbol{\mathcal{E}}] = p = M^{\epsilon-2n+1}.$$

Let $X'_i \subset X_i, i = 0, 1, \ldots, n-1$ be subsets so that $|X'_i| \geq M$. Then

$$\text{Prob}[|\boldsymbol{\mathcal{E}} \cap [\bigcup_{i\in n} X'_i]^{2n}| \leq M] \ = \ \sum_{j=0}^{M} \binom{\binom{M}{2}^n}{j} p^j (1-p)^{\binom{M}{2}^n - j} \tag{5}$$

$$\leq \ M \binom{\binom{M}{2}^n}{M} p^M (1-p)^{\binom{M}{2}^n - M} \tag{6}$$

$$
\begin{aligned}
&\sim && M(\frac{e}{2^n}M^{2n-1})^M(M^{\epsilon-2n+1})^M \exp\left(-M^{\epsilon-2n+1}\left(\binom{M}{2}^n - M\right)\right)\\
&< && M^{\epsilon M}\exp(-\frac{1}{2^n}M^{1+\epsilon})\\
&= && o(1).
\end{aligned}
$$

The first equality (5) merely sums probabilities according to the binomial distribution. In such a distribution, it is well known that the occurrence with highest probability occurs close to the expected value, which is, in this case,

$$
\binom{M}{2}^n M^{\epsilon-2n+1} = \frac{N^{1+\epsilon}}{2^n}.
$$

For large enough N, we see that this can be made larger than M, and so the inequality (6) holds. These equations show us that for sufficiently large N the graph induced by $\bigcup_{i\in n} X_i'$ has more than N/n edges with probability close to 1.

Now we will examine short cycles; if (4) fails to be true (with $r = 2n$) for some $j_0 \leq l$, then there exists an l-tuple of edges $\boldsymbol{f_0, f_1, \ldots, f_{l-1}} \in \boldsymbol{\mathcal{E}}$ and a set $Y \subset \bigcup_{i\in n} X^i$, $|Y| = l(2n-1)$ so that $\bigcup_{j\in l} \mathrm{f}_j \subset Y$. The number of choices for each set Y is bounded by

$$
\binom{nN}{l(2n-1)} < c_1 N^{l(2n-1)}
$$

and given Y, the number of choices for $\boldsymbol{f_0, f_1, \ldots, f_{l-1}}$ is easily bounded by

$$
\binom{l(2n-1)}{2n}^l < c_2
$$

where $c_1 = c_1(n,l)$, $c_2 = c_2(n,l)$ are independent of the choice of N. Thus the expected number of cycles of length at most l can be bounded from above by

$$
c_1c_2N^{l(2n-1)}p^l \leq c_1c_2N^{l(2n-1)}\left(\frac{2N}{n}\right)^{l(\epsilon-2n+1)} = c_3\sqrt{N} = o(N) = o(M),
$$

where $c_3 = c_3(n,l)$ is a constant independent of N.

Summarizing, with large probability $((X_i)_{i\in n}, \boldsymbol{\mathcal{E}})$ has the properties (i): $|[\bigcup_{i\in n} X_i']^{2n}\cap\boldsymbol{\mathcal{E}}| > M$ whenever $X_i' \subset X_i$ and $|X_i'| \geq \frac{1}{n}|X_i|$ for all $i \in n$, and (ii): the number of cycles of length at most l is $o(M)$.

Let $((X_i)_{i\in n}, \mathcal{E}')$ be a hypergraph satisfying both (i) and (ii) (while still satisfying condition 1 in the statement of the theorem). Delete an edge from each cycle of length at most l to obtain a hypergraph $E = ((X_i)_{i\in n}, \mathcal{E})$. We deleted at most $o(M)$ edges, and thus due to (i),

$$
|[\bigcup_{i\in n} X_i']^{2n} \cap \mathcal{E}| > M - o(M) > 0
$$

whenever $X_i' \subset X_i$ and $|X_i'| \geq \frac{1}{n}|X_i|$.$\square$

The hypergraph constructed in the above theorem has a very special property; with the help of this next useful lemma, we shall find it. For a given order $\leq^*$ on a set A, we use $C \leq^* D$ to denote $c \leq^* d$ for all $c \in C \subset A$ and $d \in D \subset A$, where no relations in C or D are specified.

Lemma 6.2 *For given n, $N \in \omega$, let $(X, \leq^*)$ be a totally ordered set with $|X| = nN$. Let $X = X_0 \dot\cup X_1 \dot\cup \ldots \dot\cup X_{n-1}$ be a partition of X with $|X_i| = N$ for each $i \in n$. Then there exists a subfamily $X'_0, X'_1, \ldots, X'_{n-1}$, where for each i $X'_i \subseteq X_i$ and $|X'_i| \geq N/n$, together with a permutation $\sigma : n \to n$ so that $X'_{\sigma(0)} <^* X'_{\sigma(1)} <^* \ldots <^* X'_{\sigma(n-1)}$.*

Proof: Since the case $n = 1$ is trivial, assume $n \geq 2$ and let X_i $(i \in n)$ and $(X, \leq^*)$ be given with $x_0 \leq^* x_1 \leq^* \ldots \leq^* x_{nN-1}$ an enumeration of X. First we select the smallest $k_0 \in (nN-1)$ so that for some $i \in n$, $|\{x_0, \ldots, x_{k_0-1}\} \cap X_i| = \lceil N/n \rceil$, and set $\sigma(0) = i$. Note that $k_0 \leq n(\lceil N/n \rceil - 1) + 1 \leq N$ by the pigeon hole principle. Also observe that $|\{x_{k_0}, \ldots, x_{nN-1}\} \cap X_j| > N - \lceil N/n \rceil$ for each $j \neq \sigma(0)$ since at most $\lceil N/n \rceil - 1$ elements of X_j $(j \neq \sigma(0))$ ocurred in $\{x_0, \ldots, x_{k_0-1}\}$ and k_0 was chosen smallest. We set $X'_{\sigma(0)} = \{x_0, \ldots, x_{k_0-1}\} \cap X_{\sigma(0)}$. We repeat the procedure with $\{x_{k_0}, \ldots, x_{nN-1}\}$ and $\{X_i : i \neq \sigma(0)\}$. In general, suppose we have found $J = \{\sigma(0), \ldots, \sigma(t-1)\}$ and $\{X'_j : j \in J\}$ so that $X'_{\sigma(0)} <^* \ldots <^* X'_{\sigma(t-1)}$ where $max(X'_{\sigma(t-1)}) = x_{k_{(t-1)}-1}$ with $t < n$. Then for $\nu \notin J$,

$$|\{x_{k_{t-1}}, \ldots, x_{nN-1}\} \cap X_\nu| \geq [N - t(\lceil N/n \rceil - 1)] > (n-t)(\lceil N/n \rceil - 1),$$

where the first inequality is because we could have 'used' only so many at each step and the second inequality holds since $N > n(\lceil N/n \rceil - 1)$. Thus we can continue, finding $\sigma(t) \in n \backslash J$ and a minimal k_t so that $|X_{\sigma(t)} \cap \{x_i : k_{t-1} \leq i < k_t\}| \geq \lceil N/n \rceil$ and so we set $X'_{\sigma(t)} = X_{\sigma(t)} \cap \{x_i : k_{t-1} \leq i < k_t\}$. $\square$

Let $E = ((X_i)_{i\in n}, \mathcal{E})$ be the hypergraph guaranteed by Theorem 6.1. Since for each $e \in \mathcal{E}$, $|e \cap X_i| = 2$ for each $i \in n$, let us denote each edge by $e = \{x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}\}$ where $x_i, y_i \in X_i$ for each $i \in n$.

Lemma 6.3 *For $E = ((X_i)_{i\in n}, \mathcal{E})$ and $<^*$ a total order on $\bigcup_{i\in n} X_i$, then there exists $e = \{x_0, y_0, x_1, y_1, \ldots, x_{n-1}, yn-1\} \in \mathcal{E}$ and a permutation σ of n so that the vertices of e satisfy $x_{\sigma(0)} <^* y_{\sigma(0)} <^* x_{\sigma(1)} <^* y_{\sigma(1)} <^* \ldots <^* x_{\sigma(n-1)} <^* y_{\sigma(n-1)}$, where $\{x_{\sigma(i)}, y_{\sigma(i)}\} \subset X_{\sigma(i)}$ for each $i \in n$. That is, there remains at least one edge which keeps vertices from the same coordinate X_i 'together' in the order $<^*$.*

Proof: Let $<^*$ be a given order on $\bigcup_{i\in n} X_i$; then by Lemma6.2 there exists a subfamily $X'_0, X'_1, \ldots, X'_{n-1}$, where for each i $X'_i \subseteq X_i$ and $|X'_i| \geq N/n$, and a permutation $\sigma : n \to n$ so that $X'_{\sigma(0)} <^* X'_{\sigma(1)} <^* \ldots <^* X'_{\sigma(n-1)}$. Now by condition 3 of Theorem 6.1, the desired edge exists.

7 Infinite family of counterexamples

In this section, we produce infinitely many pairs H and G so that $mdo(H, G) \geq 2$ and yet $\mathcal{R}[(G)^H_2] \neq \emptyset$. We do this by choosing H of a particular nature (of which there are infinitely many) and, using the large hypergraph of Theorem 6.1, produce a corresponding G satisfying the sought after conditions. We first give a simple observation.

Lemma 7.1 *All connected non-trivial ordinary graphs contain a copy of P_2 as an induced subgraph.*

Proof: Let $H = (V(H), E(H))$ be connected. Choose $a, b \in V(H)$ so that $\{a, b\} \notin E(H)$. Since H is connected there exist $x_1, x_2, \ldots, x_m \in V(H)$ determining a path $ax_1x_2 \ldots x_mb$. Assume that no copy of P_2 occurs as an induced subgraph of the graph induced by $\{a, x_1, x_2, \ldots, x_m\}$. Then we must have $\{a, x_2\} \in E(H)$ (otherwise a,x_1 and x_2 determine a copy of P_2). Similarly, $\{a, x_3\}, \ldots, \{a, x_m\}$ must also be edges. In this case a,x_m and b determine a copy of P_2.□

Recall that an ordinary graph is *n-connected* if between any two vertices there are n vertex-disjoint paths joining them. It is easy to see that a graph is 2-connected if and only if the graph can not be made disconnected by the removal of any single vertex, that is, its smallest *cutset* contains at least two elements. We also say, in this case, that the graph has no *cutpoints*. (A cutpoint is a vertex whose removal disconnects the graph.)

Theorem 7.2 *If H is a non-trivial 2-connected (ordinary) graph, then there exists a graph G so that $mdo(H, G) \geq 2$ and $\mathcal{R}[(G)_2^H] \neq \emptyset$.*

Proof: Let $H = (V(H), E(H))$ be given with $|V(H)| = n$. By Lemma 7.1 fix a copy of P_2 across $\{h_0, h_1, h_2\} \subset V(H)$ where we enumerate $V(H) = \{h_0, h_1, \ldots, h_{n-1}\}$. Form a new graph $K \cong H$, $V(H) \cap V(K) = \emptyset$, with $\psi : V(H) \to V(K) = \{k_o, k_1, \ldots, k_{n-1}\}$, the isomorphism defined by $\psi(h_i) = k_{i+1(\bmod 3)}$ for $i = 0$, 1 and 2 and $\psi(h_i) = k_i$ otherwise. We have simply relabeled H using a permutation of the first three vertices. Order each of $V(H)$ and $V(K)$ in the natural way (i.e., $h_i < h_j$ and $k_i < k_j$ if and only if $i < j$) producing $(H, \leq)$ and $(K, \leq)$. Note that $(H, \leq) \ncong (K, \leq)$.

Select a hypergraph $E = ((X_i)_{i\in n}, \mathcal{E})$ satisfying the conditions in Theorem 6.1 with girth$(E) > n$. Construct a new (ordinary) graph G on the vertex set $\bigcup_{i\in n} X_i$ by disjointly embedding a copy of $H\dot{\cup}K$ into each hyperedge of E in the following manner: For each hyperedge $e = \{x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}\}$ (where $x_i, y_i \in X_i$ for each $i \in n$) in $\mathcal{E}$ define embeddings $f_e : V(H)\dot{\cup}V(K) \to e$ by $f_e(h_i) = x_i$ and $f_e(k_i) = y_i$ for each $i \in n$. So $\{a, b\} \in E(G)$ if and only if $\{f_e^{-1}(a), f_e^{-1}(b)\} \in E(H) \cup E(K)$ for some $e \in \mathcal{E}$.

Since girth$(E) > 2$, hyperedges of E intersect in at most one point and so these embeddings are well defined. Essentially, $G = ((X_i)_{i\in n}, E(G))$ is a graph formed by 'stringing out' copies of H across its coordinates; the H-subgraphs can sit in one of two ways. We claim that $mdo(H, G) \geq 2$.

Let $\leq$ be an order on $V(G) = V(E)$. By Lemma 6.3, there exists a hyperedge $e \in \mathcal{E}$ which respects grouping of vertices along coordinates, and hence we find a copy of H and a copy of K which satisfy $\{f_e(h_i), f_e(k_i)\} \subset X_{\sigma(i)}$ for each i and some permutation σ. We need only notice that permuting the order of the vertices of both $(H, \leq)$ and $(K, \leq)$ in the same way produces again two non-isomorphic orderings of H. So any ordering of G produces two non-isomorphic ordered H's, i.e., $mdo(H, G) \geq 2$. We have yet to demonstrate that $\mathcal{R}[(G)_2^H] \neq \emptyset$.

Rename $(H, \leq) = (H, \leq^0)$ and $(K, \leq) = (H, \leq^1)$ and fix $(H, \leq^2)$, a third ordering of H defined by $h_2 \leq^2 h_0 \leq^2 h_1 \leq^2 h_3 \leq^2 h_4 \leq^2 \ldots \leq^2 h_{n-1}$, agreeing with the first two except by a cyclical permutation on h_0, h_1 and h_2. Let $\leq_0$, $\leq_1$ and $\leq_2$ be three total orders on $V(G)$ which preserve coordinates and agree except that the first three coordinates are permuted, e.g.:

$$X_0 \leq_0 X_1 \leq_0 X_2 \leq_0 X_3 \leq_0 \ldots \leq_0 X_{n-1},$$

$$X_1 \leq_1 X_2 \leq_1 X_0 \leq_1 X_3 \leq_1 \ldots \leq_1 X_{n-1},$$
$$X_2 \leq_2 X_0 \leq_2 X_1 \leq_2 X_3 \leq_2 \ldots \leq_2 X_{n-1}.$$

Now each of $(G, \leq_0)$, $(G, \leq_1)$ and $(G, \leq_2)$ can be seen to contain a different pair of $(H, \leq^0)$, $(H, \leq^1)$ and $(H, \leq^2)$ as induced subgraphs. We claim that these are the *only* induced H-subgraphs of the given ordered G's.

In the construction of G no edges were added between hyperedges of E. Since hyperedges of E intersect in at most one point and H is 2-connected, no new copy of H is introduced by two hyperedges intersecting. (If one was newly formed, the point of intersection would be a cutpoint, contrary to being 2-connected.) The introduction of more hyperedges intersecting the first two might help to construct a new copy of H except that the condition 'girth$(E) > n$' prevents any such occurrence. So no other copies of H exist in G other than those produced explicitly in the construction. Recall now Theorem 5.1, and using the three orderings of G and the three orderings of H, we see $\chi(K_{H,G}) \geq 3$. Hence $\mathcal{R}[(G)_2^H] \neq \emptyset$. □

We denote the complement of an ordinary graph G by $\overline{G}$. Given a collection $\mathcal{G}$ of graphs, we define $\overline{\mathcal{G}} = \{\overline{G} : G \in \mathcal{G}\}$. Let $\mathcal{M}$ be the collection of all graphs G containing a cutpoint which is connected to all other vertices except at most one. It is not difficult to derive the following:

Lemma 7.3 *$H \in \mathcal{M} \cup \overline{\mathcal{M}}$ if and only if neither H nor $\overline{H}$ is 2-connected.*

This was proved in [18]. Using this terminology, we obtain an immediate corollary of Theorem 7.2.

Corollary 7.4 *Let H be a non-trivial ordinary graph. If $H \notin \mathcal{M} \cup \overline{\mathcal{M}}$ then there exists a graph G so that $mdo(H, G) \geq 2$ and $\mathcal{R}[(G)_2^H] \neq \emptyset$.*

Proof: If $H \notin \mathcal{M} \cup \overline{\mathcal{M}}$ then either H or $\overline{H}$ is 2-connected. If H is 2-connected, we are done by Theorem 7.2. Suppose H is not 2-connected but $\overline{H}$ is. By Lemma 7.1, $\overline{H}$ contains a copy of P_2. So we duplicate the construction in Theorem 7.2 to produce G and F with $F \longrightarrow (G)_2^{\overline{H}}$. We claim that $\overline{F} \longrightarrow (\overline{G})_2^H$. Fix a coloring $\Delta : \binom{\overline{F}}{H} \longrightarrow 2$. This induces a coloring $\overline{\Delta} : \binom{F}{\overline{H}} \longrightarrow 2$ by $\overline{\Delta}(\overline{H}) = \Delta(H)$. Since $F \longrightarrow (G)_2^{\overline{H}}$ there exists a $G' \in \binom{F}{G}$ so that $\binom{G'}{\overline{H}}$ is monochromatic with respect to $\overline{\Delta}$. Hence $\binom{\overline{G'}}{H}$ is monochromatic with respect to Δ.

So we only need show $mdo(H, \overline{G}) \geq 2$. Choose an ordering $(\overline{G}, \leq)$ of $\overline{G}$. If $(H, \leq^0)$, $(H, \leq^1) \in DO(H, \overline{G}, \leq)$ are non-isomorphic, then certainly $(\overline{H}, \leq^0)$, $(\overline{H}, \leq^1) \in DO(\overline{H}, G, \leq)$ are non-isomorphic as well.□

We conclude with some remarks. Many generalizations of these results to hypergraphs are possible. The notions of connectivity and complement must extended however. One may also choose a subhypergraph which plays the role of P_2 in Theorem 7.2, but with cautionary heed to extra assumptions. Are there 'elegant' extensions of this type?

We save for another discussion a partial classification of ordinary graphs for which the statement "$mdo(H, G) = 1$ if and only if $\mathcal{R}[(G)_2^H] \neq \emptyset$" is true. Alternately, how much can the conditions on H be weakened so that this statement fails; Lemma 4.1 shows that H need not be 2-connected. Numerous other interesting questions should suggest themselves to the reader now. If one is to complete the classification of ordinary graphs with respect to Ramsey properties, it is believed that pursuits in directions similar to those taken here may be of assistance.

References

[1] F. G. Abramson and L.A. Harrington, Models without discernibles, *J. Symbolic Logic* **43** (1978),572-600.

[2] W. Deuber, Generalizations of Ramsey's theorem, in *Infinite and finite sets* (A. Hajnal et. al eds.) *Coll. Math. Soc. J. Bolyai* **10**, North Holland (1975),323-332.

[3] W. Deuber, Partitionstheoreme für Graphen, *Comment. Math. Helvetici* **50** (1975),311-320.

[4] P. Erdős, Graph theory and probability,*Canad. J. Math.* **11** (1959),34-38.

[5] P. Erdős, A. Hajnal, On chromatic number of graphs and set systems, /it Acta. Math. Acad. Sci. Hungar. **17** (1966),61-99.

[6] P. Erdős, A. Hajnal and L. Pósa, Strong embeddings of graphs into colored graphs, in *Infinite and finite sets* (A. Hajnal et. al eds.)*Coll. Math. Soc. J. Bolyai* **10**, North Holland (1975),585-595.

[7] P. Erdős, J. Spencer, *Probabilistic methods in combinatorics*, Academic Press New York, 1974.

[8] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, SIAM *J. Appl. Math.* **18** (1970),19-29.

[9] J. Nešetřil and V. Rödl, Partitions of subgraphs, in *Recent Advances in Graph Theory* (M. Fiedler ed.), Academica Praha, Prague (1975),413-423.

[10] J. Nešetřil and V. Rödl, Partitions of finite relational and set systems *J. Combin. Theory Ser. A* **22** (1977),289-312.

[11] J. Nešetřil and V. Rödl, Simple proof of the existence of restricted Ramsey graphs by means of a partite construction, *Combinatorica* **1** (2)(1981),199-202.

[12] J. Nešetřil and V. Rödl, Ramsey classes of set systems, *J. Combin. Theory Ser. A* **34** (1983),183-201.

[13] J. Nešetřil and V. Rödl, The partite construction and Ramsey set systems, *Discrete Math.* **75**, North Holland (1989),327-334.

[14] H. J. Prömel, *Ramsey theory for discrete structures*, (Manuscript, 1987).

[15] H. J. Prömel and B. Voigt, Canonizing Ramsey theorems for finite graphs and hypergraphs, *Discrete Math.* **54** (1985),49-59.

[16] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930),264-286.

[17] V. Rödl, A generalization of Ramsey theorem, in *Graphs, Hypergraphs and Block Systems*, Sielona Gora (1976),211-220.

[18] V. Rödl, N. Sauer, *The Ramsey property and families of graphs characterized by excluding a graph*, (Manuscript).