Independent Deuber sets in graphs on the natural numbers

David S. Gunderson* Imre Leader† Hans Jürgen Prömel‡ Vojtěch Rödl §

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Abstract

We show that for any k, m, p, c, if G is a K_k-free graph on $\mathbb{N}$ then there is an independent set of vertices in G that contains an (m, p, c)-set. Hence if G is a K_k-free graph on $\mathbb{N}$, then one can solve any partition regular system of equations in an independent set. This is a common generalization of partition regularity theorems of Rado (who characterized systems of linear equations $A\mathbf{x} = \mathbf{0}$ a solution of which can be found monochromatic under any finite coloring of $\mathbb{N}$) and Deuber (who provided another characterization in terms of (m, p, c)-sets and a partition theorem for them), and of Ramsey's theorem itself.

1 Introduction and statement of results

In this paper we are interested in graphs whose vertices are natural numbers, and in arithmetic properties of independent sets in such graphs. We use $\mathbb{N}$ to denote the set of natural numbers—not including 0—and $\omega = \mathbb{N} \cup \{0\}$. We write $[a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}$ to denote an interval of integers.

Let A be a finite matrix with integer entries. The system of linear equations $A\boldsymbol{x} = \mathbf{0}$ is called *partition regular* (over $\mathbb{N}$) if for every partition of $\mathbb{N}$ into finitely many classes there exists a solution completely contained in one class.

Schur's theorem [17] says that for any positive integer r, there exists n so that for every coloring $\rho : [1, n] \to [1, r]$ there exist $x, y \in [1, n]$ with $\rho(x) = \rho(y) = \rho(x + y)$. The equation $x + y - z = 0$ describes these Schur triples, and so is partition regular. Van der Waerden's theorem [19] states that for any positive integers r, ℓ, there exists n so that for any coloring $\rho : [1, n] \to [1, r]$ there is a monochromatic ℓ-term arithmetic progression. Solutions to equations $x - 2y + z = 0$ are 3-term arithmetic progressions or are constant and so this system is also partition regular. Similarly, systems of equations describing any longer arithmetic progressions

*Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada, R3T 2N2; email: *gunderso@cc.umanitoba.ca* Partially supported by NSERC grant 228064-00.

†University College London, Gower Street, London WC1E 6BT; present address: Department of Pure Mathematics and Mathematical Sciences, Wilberforce Road, Cambridge, England, CB3 0WB; email: *i.leader@dpmms.cam.ac.uk*

‡Institut für Informatik; Humboldt Universität zu Berlin; Unter den Linden 6, 10099 Berlin, Germany; e-mail: *proemel@informatik.hu-berlin.de* Partially supported by DFG grant Pr 296/4-2.

§Mathematics and Computer Science; Emory University; Atlanta, GA 30322; e-mail: *rodl@mathcs.emory.edu* Partially supported by NSF grant DMS-9704114. Part of this work was done while as an Alexander von Humboldt Senior Scientist visiting Humboldt-Universität zu Berlin.

form partition regular systems. An example of a simple system which is not partition regular is $x + y = 3z$. (See, e.g., [6] or [7] for more details.)

A characterization of partition regular systems of equations was first given by Rado [15] in terms of something (which is not relevant to our use here) called the "columns property". Deuber [2] later gave another characterization of partition regular systems using structures called "(m, p, c)-sets", which we now define.

Definition 1.1 *Let $p, c \in \mathbb{N}$ with $c \leq p$, and let $m \in \omega$. A set of integers S is an (m, p, c)-set if $S \subset \mathbb{N}$ and there exist positive integers (generators) $x_0, x_1, \ldots, x_m$ so that $S = R_0(S) \cup R_1(S) \cup \cdots \cup R_m(S)$, where*

$$\begin{array}{lcl} R_0(S) & = & \{cx_0 + \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \ \lambda_1, \ldots, \lambda_m \in [-p, p]\}, \\ R_1(S) & = & \{cx_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \ \lambda_2, \ldots, \lambda_m \in [-p, p]\}, \\ \vdots & & \vdots \\ R_{m-1}(S) & = & \{cx_{m-1} + \lambda_m x_m : \ \lambda_m \in [-p, p]\}, \\ R_m(S) & = & \{cx_m\}. \end{array}$$

In this case we write $S = (x_0, x_1, \ldots, x_m)_{p,c}$ and we say that $R_k(S)$ is the $(k+1)$-st row of S.

We note that the condition $c \leq p$ is for convenience only; nothing would be lost without this condition because for any $p' > p$, every (m, p', c)-set trivially contains an (m, p, c)-set.

In honor of Deuber's contributions to the field, if a set S is an (m, p, c)-set for some m, p, c, then we might simply say that S is a *Deuber set* without specifying the parameters m, p, c.

Theorem 1.2 (Deuber [2]) *A linear system $A\boldsymbol{x} = \mathbf{0}$ is partition regular if and only if there exist positive integers m, p, c such that every (m, p, c)-set contains a solution of $A\boldsymbol{x} = \mathbf{0}$.*

In proving a conjecture by Rado regarding partition regular systems, Deuber used the following partition theorem.

Theorem 1.3 (Deuber [2]) *For every $m \in \omega$, every $p, c \in \mathbb{N}$ with $c \leq p$ and every $k \in \mathbb{N}$, there exists $n, q, d \in \mathbb{N}$ with $d \leq q$ so that for every (n, q, d)-set X and every coloring $\rho : X \to [1, k]$, there exists a monochromatic (m, p, c)-set contained in X.*

To state our results, we adopt standard notation. For a set S and $n \in \omega$, let $[S]^n = \{F \subseteq S : |F| = n\}$. Let $G = (V, E)$ denote a (simple) graph on vertex set $V = V(G)$ with edge set $E = E(G) \subseteq [V]^2$. A set $Y \subset V(G)$ is called *independent* in G if $[Y]^2 \cap E(G) = \emptyset$. When $E(G) = [V(G)]^2$, we say that G is complete, and the complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is *k-partite* if V can be partitioned into k sets, $V = V_1 \cup \cdots \cup V_k$, each V_i containing no edges, and is a *complete k-partite* graph if for each $i \neq j$, whenever $x \in V_i$ and $y \in V_j$ then $\{x, y\} \in E$. A complete bipartite graph on sets of size m and n will be denoted by $K_{m,n}$.

The main result in this paper is the following.

Theorem 1.4 *Given $k, p, c \in \mathbb{N}$ with $c \leq p$ and $m \in \omega$, there exist $n, q, d \in \mathbb{N}$ so that any K_k-free graph on an (n, q, d)-set contains an independent (m, p, c)-set.*

Remarks: 1. Theorem 1.4 generalizes Theorem 1.3 by the following reasoning: Fix k, m, p, c, let n, q, d be guaranteed by Theorem 1.4, and fix an (n, q, d)-set X. Color X with $r = k-1$ colors and form the complete $(k-1)$-partite graph G whose partite sets are color classes. Since G is K_k-free, by Theorem 1.4 some (m, p, c)-set is independent in G, and hence must be contained in one partite set, i.e., a single color class. Hence there is a monochromatic (m, p, c)-set. Since (m, p, c)-sets contain sum-sets and arithmetic progressions, Theorem 1.4 also implies theorems of van der Waerden, Schur, and others.

2. Theorem 1.4 also generalizes Ramsey's theorem for graphs, because under any red-blue coloring of the pairs of a large set, rather than guaranteeing either a red K_k or a large blue clique, we guarantee either a red K_k, or a large blue clique on an (m, p, c)-set.

Since any (n, q, d)-set sits in some initial interval of the positive integers, Theorem 1.4 immediately implies the following statement:

Corollary 1.5 *Given $k, p, c \in \mathbb{N}$, $m \in \omega$, and any K_k-free graph G with vertex set $\mathbb{N}$, there exists an (m, p, c)-set which is independent in G.*

Corollary 1.5 can be formulated in terms of partition regular systems by using the m, p, c guaranteed in Theorem 1.2:

Corollary 1.6 *For any $k \geq 2$ and any K_k-free graph on $\mathbb{N}$, one can solve any partition regular system in an independent set.*

We do not know if there is a hypergraph version of Theorem 2.1. For example, is there an analogous condition on a family of triples of $\mathbb{N}$ that would imply that there is an (m, p, c)-set not containing any triple? If so, it is not so simple, as the following example indicates. Let H be the 3-uniform hypergraph on $\mathbb{N}$ defined with hyperedges of the form $\{x, x+d, x+3d\}$. Then H is $K_4^{(3)}$-free, yet every arithmetic progression of length 4 contains a hyperedge.

2 Earlier work

Ramsey's theorem for graphs [16] says that for any positive integers r, and m, there exists n so that for any coloring $\rho : [1, n]^2 \to [1, r]$, there exists $M \in [1, n]^m$ so that $[M]^2$ is monochromatic. Erdős [4] asked whether the following natural generalization of both Ramsey's theorem and Schur's theorem holds: If G is a triangle-free graph on vertex set $\mathbb{N}$, does there always exist an independent Schur triple, that is, do there exist x, y, $x \neq y$ so that $\mathrm{FS}(x, y) = \{x, y, x+y\}$ is independent in G? The answer is yes, as proved in [13] where it was shown that in fact, for fixed k and d, if G is a K_k-free graph on $\mathbb{N}$, then there exist distinct integers $a_1, a_2, ..., a_d$, so that the finite sum set $\mathrm{FS}(\{a_1, \ldots, a_d\})$ is an independent set in G. Harborth *et al.* (see, e.g., [1], [11]) have given some sharp lower bounds on n so that if G is a graph on $[1, n]$, these results hold (except the a_i need not be distinct).

Related progress was also made for an infinite version of Erdős' question. Given a set $\{x_i\}_{i \in I}$ of distinct positive integers, let $\mathrm{FS}(\{x_i\}_{i \in I}) = \left\{\sum_{j \in J} x_j : \emptyset \neq J \subseteq I, |J| < \infty\right\}$ denote the finite sums (with no repetitions) from the set. When I is infinite, we say that $\mathrm{FS}(\{x_i\}_{i \in I})$ is a *Hindman set*. In 1995, Hajnal asked the following (see [5]): If G is a triangle-free graph on $\mathbb{N}$, does there always exist a Hindman set independent in G? Hajnal's question has been answered in the negative in [3]. Variants of Hajnal's question have been shown to indeed have a positive answer; for example, if the condition "triangle-free" is replaced by "$K_{k,k}$-free" (see [3], [9], and [13]).

A common generalization of Ramsey's theorem and van der Waerden's theorem was also found in [9]: For fixed k and ℓ, if G is a K_k-free graph on $\mathbb{N}$, then there exists an ℓ-term arithmetic progression which spans an independent set in G.

Coloring theorems for arithmetic progressions or finite sums have abstract analogues (the Hales-Jewett theorem and the Graham-Rothschild theorem, respectively; see, e.g., [7] or [14]), from which they can be deduced instantly. In contrast, Deuber's theorem for partitioning (m,p,c)-sets can not be accomplished by any one application of such a theorem; several iterations are required. As with Deuber's theorem, one would not expect to be able to prove our main result for (m,p,c)-sets with any single application of an abstract theorem. Indeed, the first situation where a single process does not seem to work is for an arithmetic progression together with its difference—which is the simplest kind of subset of an (m,p,c)-set not necessarily contained in any one row. In [9] it was proved that for any $k,\ell \geq 3$, in any K_k-free graph, there exists an ℓ-term arithmetic progression together with its difference, all contained in an independent set. This proof used a form of the Gallai-Witt theorem applied iteratively; it does not seem to follow from one application of any of the major abstract theorems (like the Hales-Jewett or Graham-Rothschild theorems).

3 Preliminary results

We now briefly describe one of our main tools, the Hales-Jewett theorem.

Let A denote a finite alphabet; write $A^s = \{(x_1,\ldots,x_s) : x_i \in A\}$. Let $[1,s] = F \cup M_1 \cup \cdots \cup M_t$ be a partition with $|M_j| > 0$ for $j = 1,2,\ldots,t$ and let $(g_i)_{i\in F} \in A^F$ be a fixed $|F|$-tuple. A *t-dimensional subcube* of A^s (associated with $(g_j)_{j\in F}$ and the partition $F \cup M_1 \cup \cdots \cup M_t$) is a set of the form

$$\begin{aligned}\mathrm{HJC}(F, M_1,\ldots,M_t,(g_j)_{j\in F}) = \{(x_1,\ldots,x_s) \quad : \quad & x_j = g_j \text{ for } j \in F \text{ and} \\ & x_j = x_{j'} \text{ if } j,j' \in M_\alpha \text{ for some } \alpha\}.\end{aligned}$$

We now state the central theorem regarding parameter sets.

Theorem 3.1 (Hales-Jewett) *For every $t,r \in \mathbb{N}$ and every finite alphabet A, there exists $s = HJ(t,r,|A|)$ so that for every coloring $\rho : A^s \to [1,r]$, there exists a monochromatic t-dimensional subcube of A^s.*

The original version in [10] yields a 1-dimensional subcube. That version easily implies the current version (see [7, p. 40]). See [14] for a survey of results, applications, and notation for parameter words, another language to describe the Hales-Jewett theorem.

Definition 3.2 *Let $p,c,q,d \in \mathbb{N}$ with $c \leq p$ and $q \leq d$, let $U = (x_0,x_1,\ldots,x_m)_{p,c}$ be an (m,p,c)-set, and let $V = (y_0,y_1,\ldots,y_n)_{q,d}$ be an (n,q,d)-set. We say that U is naturally contained in V, written $U \preceq V$ if and only if there is a strictly increasing function*

$$\psi : \{0,1,\ldots,m\} \to \{0,1,\ldots,n\}$$

such that for each $i \in \{0,1,\ldots,m\}$, $R_i(U) \subseteq R_{\psi(i)}(V)$.

Notice that natural containment is trivially transitive.

Example 3.3 *The* $(1,3,1)$*-set* $A=(10,1)_{3,1}$ *is contained in the* $(2,2,1)$*-set* $B=(20,5,1)_{2,1}$ *but* A *is not naturally contained in* B*. In fact,* $R_0(A)$ *is not contained in any row of* B*.*

We now present some simple results that guarantee that all inclusions with which we shall be concerned are natural.

For any $p,m\in\mathbb{N}$, we use

$$[a_0,a_1\ldots,a_m]_p=\{\lambda_0a_0+\lambda_1a_1+\ldots+\lambda_ma_m:\lambda_0,\lambda_1,\ldots,\lambda_m\in[-p,p]\},$$

to denote the span of $a_0,a_1,\ldots,a_nm$.

Lemma 3.4 *Let* $m,p,x_0,x_1,\ldots,x_m\in\mathbb{N}$*. If for each* $i\in\{0,1,\ldots,m-1\}$*,* $x_i>2p\sum_{j=i+1}^m x_j$*, then expressions in* $[x_0,x_1,\ldots,x_m]_p$ *are unique. That is, if*

$$\lambda_0,\lambda_1,\ldots,\lambda_m,\mu_0,\mu_1,\ldots,\mu_m\in[-p,p]\ \textit{and}\ \sum_{i=1}^m\lambda_ix_i=\sum_{i=1}^m\mu_ix_i\,,$$

then for each $i\in\{0,1,\ldots,m\}$*,* $\lambda_i=\mu_i$*.*

Proof: Assume that $\lambda_0,\lambda_1,\ldots,\lambda_m,\mu_0,\mu_1,\ldots,\mu_m\in[-p,p]$ and $\sum_{i=1}^m\lambda_ix_i=\sum_{i=1}^m\mu_ix_i$. Suppose that there is some $i\in\{0,1,\ldots,m\}$ such that $\lambda_i\neq\mu_i$ and pick the first such i. Assume without loss of generality that $\lambda_i>\mu_i$. If $\lambda_mx_m=\mu_mx_m$, then $\lambda_m=\mu_m$, so we have that $i<m$. Then

$$\sum_{j=i+1}^m(\mu_j-\lambda_j)x_j=(\lambda_i-\mu_i)x_i\geq x_i>2p\sum_{j=i+1}^m x_j\geq\sum_{j=i+1}^m(\mu_j-\lambda_j)x_j,$$

a contradiction. □

With a little more work one can show that $6pc$ can be replaced by $5pc$ in the following lemma. (Here and later, if $i=m$, then we set $\sum_{j=i+1}^m x_j=0$.)

Lemma 3.5 *Let* $m,p,c\in\mathbb{N}$ *with* $c\leq p$ *and assume that* $x_0,x_1,\ldots,x_m\in\mathbb{N}$ *and* $(x_0,x_1,\ldots,x_m)_{6pc,c}\subseteq\mathbb{N}$*. Let* $U=(x_0,x_1,\ldots,x_m)_{p,c}$*. For each* $i\in\{0,1,\ldots,m-1\}$*,* $x_i>6p\sum_{j=i+1}^m x_j$ *and* $\min R_i(U)>\max R_{i+1}(U)$*. Also, any length 3 arithmetic progression in* U *is contained in some row of* U*.*

Proof: Let $i\in\{0,1,\ldots,m-1\}$. Then $cx_i-\sum_{j=i+1}^m 6pcx_j\in\mathbb{N}$ so $x_i>6p\sum_{j=i+1}^m x_j$. Thus

$$\min R_i(U)=cx_i-\sum_{j=i+1}^m px_j\geq x_i-\sum_{j=i+1}^m px_j>\sum_{j=i+1}^m px_j\geq cx_{i+1}+\sum_{j=i+2}^m px_j=\max R_{i+1}(U).$$

Now assume we have $d>0$ such that $\{a,a+d,a+2d\}\subseteq U$. Pick $i,k\in\{0,1,\ldots,m\}$ and $\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_m,\mu_{k+1},\mu_{k+2},\ldots,\mu_m\in[-p,p]$ such that

$$\begin{aligned}a&=cx_i+\textstyle\sum_{j=i+1}^m\lambda_jx_j\text{ and}\\a+d&=cx_k+\textstyle\sum_{j=k+1}^m\mu_jx_j\,.\end{aligned}$$

Since $a+d>a$, we have that $k \le i$. Suppose that $k<i$. Then

$$\begin{array}{rcl} d &=& cx_k + \sum_{j=k+1}^{i-1} \mu_j x_j + (\mu_i - c)x_i + \sum_{j=i+1}^{m} (\mu_j - \lambda_j)x_j \text{ and so} \\ a+2d &=& 2cx_k + \sum_{j=k+1}^{i-1} 2\mu_j x_j + (2\mu_i - c)x_i + \sum_{j=i+1}^{m} (2\mu_j - \lambda_j)x_j\,. \end{array}$$

Since the absolute value of each coefficient in the expansion of $a+2d$ is at most $3p$, we have by Lemma 3.4 that $a+2d \notin U$.

Thus $k=i$ and

$$\begin{array}{rcl} d &=& \sum_{j=i+1}^{m} (\mu_j - \lambda_j)x_j \text{ and so} \\ a+2d &=& cx_i + \sum_{j=i+1}^{m} (2\mu_j - \lambda_j)x_j\,. \end{array}$$

Again by Lemma 3.4 we have that $a+2d \in R_i(U)$. □

Lemma 3.6 *Let $n,q,d \in \mathbb{N}$ with $d \le q$ and assume that $x_0,x_1,\ldots,x_n \in \mathbb{N}$ and $(x_0,x_1,\ldots,x_n)_{6qd,d} \subseteq \mathbb{N}$. If $m \in \omega$, $p,c \in \mathbb{N}$ with $c \le p$, $y_0,y_1,\ldots,y_m \in \mathbb{N}$, and $Y = (y_0,y_1,\ldots,y_m)_{p,c} \subseteq (x_0,x_1,\ldots,x_n)_{q,d} = X$, then for each $i \in \{0,1,\ldots,m\}$ there exists $j \in \{0,1,\ldots,n\}$ such that $R_i(Y) \subseteq R_j(X)$.*

Proof: We proceed by induction on m. The case $m=0$ is trivial, so assume that $m \in \mathbb{N}$ and the assertion is true for $m-1$. Let $i \in \{0,1,\ldots,m\}$. If $i>0$, then $R_i(Y) = R_{i-1}((y_1,y_2,\ldots,y_m)_{p,c})$ so the conclusion holds by the induction hypothesis. So assume that $i=0$. Let $D = R_0((y_0,y_1,\ldots,y_{m-1})_{p,c})$ and pick by the induction hypothesis some $j \in \{0,1,\ldots,n\}$ such that $D \subseteq R_j(X)$. Now let $\lambda_1,\lambda_2,\ldots,\lambda_m \in [-p,p]$. If $\lambda_m = 0$, then $cy_0 + \sum_{l=1}^{m} \lambda_l y_l \in D \subseteq R_j(X)$, so assume $\lambda_m \ne 0$. Then $E = \{cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l - \lambda_m y_m, cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l, cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l + \lambda_m y_m\}$ is a three term arithmetic progression in X so is contained $R_k(X)$ for some k by Lemma 3.5. Since $D \cap E \ne \emptyset$, $k=j$. □

Lemma 3.7 *Let $n,q,d \in \mathbb{N}$ with $d \le q$ and assume that $x_0,x_1,\ldots,x_n \in \mathbb{N}$ and $(x_0,x_1,\ldots,x_n)_{6qd,d} \subseteq \mathbb{N}$. If $m \in \omega$, $p,c \in \mathbb{N}$ with $c \le p$, $y_0,y_1,\ldots,y_m \in \mathbb{N}$, and $Y = (y_0,y_1,\ldots,y_m)_{p,c} \subseteq (x_0,x_1,\ldots,x_n)_{q,d} = X$, then $Y \preceq X$.*

Proof: Let $i \in \{0,1,\ldots,m-1\}$ and pick by Lemma 3.6 $j,k \in \{0,1,\ldots,n\}$ such that $R_i(Y) \subseteq R_j(X)$ and $R_{i+1}(Y) \subseteq R_k(X)$. We show that $j<k$. Pick $\lambda_{j+1},\lambda_{j+2},\ldots,\lambda_n,\mu_{j+1},\mu_{j+2},\ldots,\mu_n,$ $\gamma_{k+1},\gamma_{k+2},\ldots,\gamma_n \in [-q,q]$ such that

$$\begin{array}{rcl} cy_i &=& dx_j + \sum_{l=j+1}^{n} \lambda_l x_l\,, \\ cy_i + cy_{i+1} &=& dx_j + \sum_{l=j+1}^{n} \mu_l x_l, \text{ and} \\ cy_{i+1} &=& dx_k + \sum_{l=k+1}^{n} \gamma_l x_l\,. \end{array}$$

Then $cy_{i+1} = \sum_{l=j+1}^{n} (\mu_l - \lambda_l)x_l$, so by Lemma 3.4 $k \ge j+1$. □

We shall refer later to the conclusion of the following theorem by stating that "all inclusions in $(x_0,x_1,\ldots,x_n)_{q,d}$ are natural".

Theorem 3.8 *Let $n,q,d \in \mathbb{N}$ with $d \le q$ and assume that $x_0,x_1,\ldots,x_n \in \mathbb{N}$ and $(x_0,x_1,\ldots,x_n)_{6qd,d} \subseteq \mathbb{N}$. If $m,M \in \omega$, $p,c,P,C \in \mathbb{N}$ with $c \le p$ and $C \le P$, $y_0,y_1,\ldots,y_m,z_0,$ $z_1,\ldots,z_M \in \mathbb{N}$, $Y = (y_0,y_1,\ldots,y_m)_{p,c} \subseteq (x_0,x_1,\ldots,x_n)_{q,d} = X$, $Z = (z_0,z_1,\ldots,z_M)_{P,C} \subseteq X$, and $Y \subseteq Z$, then $Y \preceq Z$.*

Proof: By Lemma 3.7 we have that $Y \preceq X$ and $Z \preceq X$. We show first that for each $i \in \{0, 1, \ldots, m\}$ there exists $j \in \{0, 1, \ldots, M\}$ such that $R_i(Y) \subseteq R_j(Z)$. Suppose instead that one has $i \in \{0, 1, \ldots, m\}$, $a, b \in R_i(Y)$, and $j < k$ in $\{0, 1, \ldots, M\}$ such that $a \in R_j(Z)$ and $b \in R_k(Z)$. Since $Z \preceq X$, we have $u < v$ in $\{0, 1, \ldots, n\}$ such that $a \in R_u(X)$ and $b \in R_v(X)$. By Lemma 3.5, $R_u(X) \cap R_v(X) = \emptyset$ and by Lemma 3.6, $R_i(Y) \subseteq R_u(X)$, a contradiction.

Now let $i \in \{0, 1, \ldots, m-1\}$ and pick $j, k \in \{0, 1, \ldots, M\}$ such that $R_i(Y) \subseteq R_j(Z)$ and $R_{i+1}(Y) \subseteq R_k(Z)$. Pick $u < v$ in $\{0, 1, \ldots, n\}$ such that $R_i(Y) \subseteq R_u(X)$ and $R_{i+1}(Y) \subseteq R_v(X)$. Then $R_j(Z) \cap R_u(X) \neq \emptyset$ so $R_j(Z) \subseteq R_u(X)$. Likewise $R_k(Z) \subseteq R_v(X)$, and therefore $j < k$. $\square$

We shall need a slightly strengthened version of Deuber's Theorem (Theorem 1.3).

Theorem 3.9 *For every $m \in \omega$, every $p, c \in \mathbb{N}$ with $c \leq p$ and every $k \in \mathbb{N}$, there exists $n, q, d \in \mathbb{N}$ with $d \leq q$ so that for every (n, q, d)-set X and every coloring $\rho : X \to [1, k]$, there exists a monochromatic (m, p, c)-set naturally contained in X.*

Proof: Pick n', q', d' as guaranteed by Theorem 1.3. Let $n = n'$, $q = 6q'd'$, and $d = d'$. Let $X = (x_0, x_1, \ldots, x_n)_{q,d}$ be k-colored. Then $Y = (x_0, x_1, \ldots, x_{n'})_{q',d'}$ is naturally contained in X and by Theorem 1.3 Y contains a monochromatic (m, p, c)-set. By Lemma 3.7, this inclusion is natural. $\square$

The following technical lemma completes our preliminaries.

Lemma 3.10 *Let $m, p, c, M, P, C \in \mathbb{N}$ with $c \leq p$ and $C \leq P$. Let $w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_M \in \mathbb{N}$. If $(w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_M)_{P,C}$, then $[w_1, w_2, \ldots, w_m]_p \subseteq [v_1, v_2, \ldots, v_M]_{(2c+p)P}$.*

Proof: First consider any $\sum_{j=2}^{m} \lambda_j w_j$ with each $\lambda_j \in [-p, p]$. Then

$$cw_1 + \sum_{j=2}^{m} \lambda_j w_j \in (w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_M)_{P,C} \subseteq [cv_1, cv_2, \ldots, cv_M]_P$$

and $cw_1 \in [cv_1, cv_2, \ldots, cv_M]_P$, so $\sum_{j=2}^{m} \lambda_j w_j \in [cv_1, cv_2, \ldots, cv_M]_{2P} \subseteq [v_1, v_2, \ldots, v_M]_{2cP}$. Also $cw_1 \in [cv_1, cv_2, \ldots, cv_M]_P$ so $w_1 \in [v_1, v_2, \ldots, v_M]_P$, so for any $\lambda_1 \in [-p, p]$ one has $\lambda_1 w_1 \in [v_1, v_2, \ldots, v_M]_{pP}$ and thus $\sum_{j=1}^{m} \lambda_j w_j \in [v_1, v_2, \ldots, v_M]_{pP+2cP}$. $\square$

4 Main proof: existence of independent (m, p, c)-sets

In the proof of Theorem 1.4, we use the following earlier result.

Theorem 4.1 *For every $k, n, q, d \in \mathbb{N}$ there exist $n', q', d' \in \mathbb{N}$ so that for any (n', q', d')-set X and any K_k-free graph G with vertex set X, there exists an (n, q, d)-set S naturally contained in X, each of whose rows is an independent set in G.*

The proof of Theorem 4.1 (see [8]) is accomplished by repeating the standard parameter sets proof of Deuber's partition theorem (see, e.g., [12]), once one knows that in a K_k-free graph on a large-dimensional Hales-Jewett cube there is always a line (or, more generally, a d-dimensional subspace) that is independent— this latter fact is proved in [9].

In view of Theorem 4.1, it is sufficient to prove Theorem 1.4 under the assumption that the graph G on an (n, q, d)-set S has all rows as independent sets. Rather than prove Theorem 1.4 with this additional assumption, we will prove a stronger statement, Theorem 4.3, below. Since a large complete k-partite graph contains many copies of K_k, Theorem 1.4 will clearly follow. This somewhat stronger theorem turns out to be easier to prove.

Definition 4.2 *Let $k, p, c, t \in \mathbb{N}$ with $c \leq p$ and let $m \in \omega$. Then $\varphi(k, m, p, c, t)$ is the statement "there exist $n, q, d \in \mathbb{N}$ such that whenever S is an (n, q, d)-set and G is a graph on S such that the rows of S are independent, there exist either*

(a) an independent (m, p, c)-set contained in S or

(b) $z_1, z_2, \ldots, z_k, a_0, a_1, \ldots, a_t \in \mathbb{N}$ such that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq S$ and the sets $\langle cz_i + [a_0, a_1, \ldots, a_t]_p\rangle_{i=1}^k$ form a complete k-partite subgraph of G."

Theorem 4.3 *For all $k, p, c, t \in \mathbb{N}$ with $c \leq p$ and all $m \in \omega$, the statement $\varphi(k, m, p, c, t)$ holds.*

Proof: The proof is by induction on m and k.

For the base cases, note that for all m', p, c, t, part (b) of $\varphi(1, m', p, c, t)$ holds, and for all k', p, c, t, part (a) of $\varphi(k', 0, p, c, t)$ holds.

So assume that $k \geq 2$ and $m \geq 1$, and for the induction hypotheses, suppose that for all m', p, c, t, statement $\varphi(k-1, m', p, c, t)$ holds, and for all p, c, t, statement $\varphi(k, m-1, p, c, t)$ holds. We need to show that $\varphi(k, m, p, c, t)$ holds for all p, c, t. Pick p, c, t such that part (a) of $\varphi(k, m, p, c, t)$ fails; we show that part (b) holds.

(1) Let (M, P, C) be the (n, q, d) guaranteed by $\varphi(k, m-1, p, c, t)$.

(2) Let (N, Q, D) be such that whenever an $(N-1, Q, D)$-set is $(k-1)$-colored, it naturally contains a monochromatic $(M, (p+2c)P, cC)$-set. (Such (N, Q, D) exist by Theorem 3.9.) We may assume that $N > m$.

(3) Set $Q' = (2cC + (2c+p)P)Q$, $D' = cCD$, and let

$$T = \mathrm{HJ}(t+1, (2Q'+1)^{Nk}, (4pQ'+1)^{N+1}) + N.$$

(4) Put $Q'' = 2cp(Q')^2(N+1)$, $D'' = c^2(D')^2$, and let (n', q', d') satisfy $\varphi(k-1, N, Q'', D'', T)$.

(5) Pick (n, q, d) such that any (n, q, d)-set S' naturally contains an (n', q', d')-set for which all inclusions are natural. (Such (n, q, d) exists by Theorem 3.8.)

Since part (a) of $\varphi(k, m, p, c, t)$ fails, pick an (n, q, d)-set S' and a graph G' on S' for which the rows are independent but S' does not contain an independent (m, p, c)-set.

Pick an (n', q', d')-set S which is naturally contained in S' and such that all inclusions within S are natural. Let G be the subgraph of G' induced on S. Note that S does not contain an independent (m, p, c)-set.

Claim 1 *If $U = (v_0, v_1, \ldots, v_{M+1})_{(p+2c)P,cC} \subseteq S$, then there is an edge of G between a point in the first row of U and a point in some later row of U.*

Proof: We have $(cv_1, cv_2, \ldots, cv_{M+1})_{P,C} \subseteq U \subseteq S$ so by the choice of (M, P, C), pick an independent $(m-1, p, c)$-set $(w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_{M+1})_{P,C}$. (If clause (b) of the definition of $\varphi(k, m-1, p, c, t)$ applied, one would have $\varphi(k, m, p, c, t)$.)

Let $V = (Cv_0, w_1, w_2, \ldots, w_m)_{p,c}$. One has immediately that

$$\bigcup_{i=1}^{m} R_i(V) = (w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_{M+1})_{P,C} \subseteq \bigcup_{i=1}^{M+1} R_i(U) \subseteq S\,.$$

Also, $R_0(V) = cCv_0 + [w_1, w_2, \ldots, w_m]_p \subseteq cCv_0 + [v_1, v_2, \ldots, v_m]_{(2c+p)P} = R_0(U)$, where the inclusion holds by Lemma 3.10.

Now $R_0(V)$ is contained in a row of S, so is independent, and

$$\bigcup_{i=1}^{m} R_i(V) = (w_1, w_2, \ldots, w_m)_{p,c}\,,$$

which is independent, so, since V is not independent, there must be an edge between a point of $R_0(V)$ and a later row of V and hence between a point of $R_0(U)$ and a later row of U. □

Claim 2 *Let $w_1, w_2, \ldots, w_N \in \mathbb{N}$. Recall that $Q' = (2cC + (2c+p)P)Q$ and $D' = cCD$. Assume that for each $i \in \{1, 2, \ldots, k-1\}$, $\beta_i \in \mathbb{N}$ and $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'} \subseteq S$. Then there is some $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ (and therefore some $x \in (w_1, w_2, \ldots, w_N)_{Q',D'}$) such that for each $i \in \{1, 2, \ldots, k-1\}$ there is an edge from x to a point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$.*

Proof: Suppose not and color $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ by the first $i \in \{1, 2, \ldots, k-1\}$ such that there is no edge from x to a point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$. By the choice of (N, Q, D) pick $i \in \{1, 2, \ldots, k-1\}$ and $v_0, v_1, \ldots, v_m$ such that $(v_0, v_1, \ldots, v_M)_{(p+2c)P,cC} \subseteq (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ and for each $x \in (v_0, v_1, \ldots, v_M)_{(p+2c)P,cC}$ there is no edge from x to any point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$.

Let $U = (D\beta_i, v_0, v_1, \ldots, v_M)_{(p+2c)P,cC}$. Now

$$(v_0, v_1, \ldots, v_M)_{(p+2c)P,cC} \subseteq (w_1, w_2, \ldots, w_N)_{cCQ,cCD}\,.$$

We claim that $R_0(U) \subseteq R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$. To see this, let $y = cCD\beta_i + \sum_{l=0}^{M} \lambda_l v_l$ where each $\lambda_l \in [-(p+2c)P, (p+2c)P]$. By Lemma 3.10,

$$\sum_{l=0}^{M} \lambda_l v_l \in [w_1, w_2, \ldots, w_N]_{(2cC+(p+2c)P)Q}\,,$$

so $y \in R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$ as claimed.

Thus by Claim 1, there is an edge between a point $y \in R_0(U)$ and some point x in a later row of U. But then $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ and $y \in R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$, a contradiction. □

We now observe that there is no independent (N, Q'', D'')-set in S. Indeed, assume one has $(x_0, x_1, \ldots, x_N)_{Q'',D''} \subseteq S$. Then, since $m < N$ and $pc(D')^2 < Q''$, one has that

$$(c(D')^2 x_0, c(D')^2 x_1, \ldots, c(D')^2 x_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_N)_{Q'',D''}\,.$$

Since $(c(D')^2 x_0, c(D')^2 x_1, \ldots, c(D')^2 x_m)_{p,c}$ is not independent, neither is $(x_0, x_1, \ldots, x_N)_{Q'',D''}$.

By the choice of (n', q', d'), since there is no independent (N, Q'', D'')-set in S, pick $b_0, b_1, \ldots, b_T, z'_1, z'_2, \ldots, z'_{k-1}$ in $\mathbb{N}$ such that $(b_0, b_1, \ldots, b_T)_{Q'',D''} \subseteq S$ and the sets

$$\langle D'' z'_i + [b_0, b_1, \ldots, b_T]_{Q''} \rangle_{i=1}^{k-1}$$

form a complete $(k-1)$-partite graph.

Now let $y_0, y_1, \ldots, y_N \in [b_{N+1}, b_{N+2}, \ldots, b_T]_{2pQ'}$ and $i \in \{1, 2, \ldots, k-1\}$, and for $j \in \{0, 1, \ldots, N\}$, pick $\lambda_{j,N+1}, \lambda_{j,N+2}, \ldots, \lambda_{j,T} \in [-2pQ', 2pQ']$ such that $y_j = \sum_{l=N+1}^{T} \lambda_{j,l} b_l$. We claim that the first row of $(c^2 D' z'_i + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q',D'}$ is contained in $D'' z'_i + [b_0, b_1, \ldots, b_T]_{Q''}$. To see this, let $\mu_1, \mu_2, \ldots, \mu_N \in [-Q', Q']$, so that

$$w = c^2 (D')^2 z'_i + c^2 (D')^2 b_0 + c D' y_0 + \sum_{j=1}^{N} c^2 D' \mu_j b_j + \sum_{j=1}^{N} c \mu_j y_j$$

is a typical member of the first row of $(c^2 D' z'_i + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q',D'}$. For each $j \in \{0, 1, \ldots, N\}$, the absolute value of the coefficient of b_j in the given expansion of w is at most $c^2 D' Q' < Q''$. And for $l \in \{N+1, N+2, \ldots, T\}$, the absolute value of the coefficient of b_l in the given expansion of w is

$$|c D' \lambda_{0,l} + \sum_{j=1}^{N} c \mu_j \lambda_{j,l}| \leq c D' 2pQ' + 2pc (Q')^2 N < Q''.$$

Next we claim that $(c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q',D'}$ is contained in $(b_0, b_1, \ldots, b_T)_{Q'',D''}$. To see this, let $\mu_1, \mu_2, \ldots, \mu_N \in [-Q', Q']$ such that,

$$\text{if } r = \min\{j \in \{1, 2, \ldots, N\} : \mu_j \neq 0\}, \text{ then } \mu_r = D',$$

and let

$$w = \sum_{j=1}^{N} \mu_j (c^2 D' b_j + c y_j) = \sum_{j=1}^{N} \mu_j c^2 D' b_j + \sum_{l=N+1}^{T} \sum_{j=1}^{N} c \mu_j \lambda_{j,l} b_l.$$

Then $\mu_r c^2 D' = D''$, and for $j \in \{r+1, r+2, \ldots, N\}$, if any, the absolute value of the coefficient of b_j in the given expansion of w is at most $c^2 D' Q' < Q''$. Also, for $l \in \{N+1, N+2, \ldots, T\}$, the absolute value of the coefficient of b_l in the given expansion of w is at most $\sum_{j=r}^{N} c2p(Q')^2 < Q''$.

In particular, we have established that

$$(c^2 D' z'_i + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q',D'} \subseteq S,$$

so we may apply Claim 2.

We define $\tau : ([-2pQ', 2pQ']^{N+1})^{T-N} \to ([-Q', Q']^N)^k$ as follows: Let

$$\overline{\lambda} = ((\lambda_{0,N+1}, \lambda_{1,N+1}, \ldots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \lambda_{1,N+2}, \ldots, \lambda_{N,N+2}),$$
$$\ldots, (\lambda_{0,T}, \lambda_{1,T}, \ldots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}.$$

For $j \in \{0, 1, \ldots, N\}$, let $y_j = \sum_{l=N+1}^{T} \lambda_{j,l} b_l$. Then by Claim 2 applied to

$$\langle (c^2 D' z'_i + c^2 D' b_0 + c y_0, c^2 D' b_1 + c y_1, c^2 D' b_2 + c y_2, \ldots, c^2 D' b_N + c y_N)_{Q',D'} \rangle_{i=1}^{k-1}$$

there is some point in $(c^2D'b_1+cy_1, c^2D'b_2+cy_2, \ldots, c^2D'b_N+cy_N)_{Q',D'}$ with an edge to a point in the first row of each $(c^2D'z_i'+c^2D'b_0+cy_0, c^2D'b_1+cy_1, c^2D'b_2+cy_2, \ldots, c^2D'b_N+cy_N)_{Q',D'}$. That is there is some

$$\overline{\gamma} = ((\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{1,N}), (\gamma_{2,1}, \gamma_{2,2}, \ldots, \gamma_{2,N}), \ldots, (\gamma_{k,1}, \gamma_{k,2}, \ldots, \gamma_{k,N})) \in ([-Q, Q]^N)^k$$

such that, if $r = \min\{j \in \{1, 2, \ldots, n\} : \gamma_{k,j} \neq 0\}$, then $\gamma_{k,r} = D'$ and for each $i \in \{1, 2, \ldots, k-1\}$, there is an edge between $\sum_{j=1}^{N} \gamma_{k,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$ and

$$c^2(D')^2z_i' + c^2(D')^2b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N} \gamma_{i,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)\,.$$

Define $\tau(\overline{\lambda}) = \overline{\gamma}$.

Now since $T = HJ(t+1, (2Q'+1)^{Nk}, (4pQ'+1)^{N+1}) + N$, Pick $F, M_0, M_1, \ldots, M_t$, $\langle(\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l})\rangle_{l\in F}$,

$$\overline{\eta} = ((\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{1,N}), (\eta_{2,1}, \eta_{2,2}, \ldots, \eta_{2,N}), \ldots, (\eta_{k,1}, \eta_{k,2}, \ldots, \eta_{k,N})) \in ([-Q, Q]^N)^k\,,$$

and $r \in \{1, 2, \ldots, N\}$ such that

(1) $F, M_0, M_1, \ldots, M_t$ are pairwise disjoint;

(2) $F \cup M_0 \cup M_1 \cup \ldots \cup M_t = \{N+1, N+2, \ldots, T\}$;

(3) each $M_s \neq \emptyset$ and $\min M_0 < \min M_1 < \ldots < \min M_t$;

(4) for each $l \in F$, $(\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l}) \in [-2pQ', 2pQ']^{N+1}$;

(5) $r = \min\{j \in \{1, 2, \ldots, N\} : \eta_{k,j} \neq 0\}$ and $\eta_{k,r} = D'$; and,

(6) whenever

$$\overline{\lambda} = ((\lambda_{0,N+1}, \lambda_{1,N+1}, \ldots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \lambda_{1,N+2}, \ldots, \lambda_{N,N+2}),$$
$$\ldots, (\lambda_{0,T}, \lambda_{1,T}, \ldots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}$$

satisfies

(a) for each $l \in F$, $(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\nu_{0,l}, \nu_{1,l}, \ldots, \nu_{N,l})$ and

(b) for each $s \in \{0, 1, \ldots, t\}$ and each $l, v \in M_s$,
$(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\lambda_{0,v}, \lambda_{1,v}, \ldots, \lambda_{N,v})$

one has $\tau(\overline{\lambda}) = \overline{\eta}$, and consequently, for each $i \in \{1, 2, \ldots, k-1\}$, there is an edge between $\sum_{j=1}^{N} \eta_{k,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$ and

$$c^2(D')^2z_i' + c^2(D')^2b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l).$$

Now for each $s \in \{0, 1, \ldots, t\}$, let $a_s = \sum_{l \in M_s} (D')^2 c b_l$. For $i \in \{1, 2, \ldots, k-1\}$, let

$$z_i = (D')^2 c z_i' + (D')^2 c b_0 + \sum_{l \in F} D' \nu_{0,l} b_l + \sum_{j=1}^{N} D' c \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{i,j} \nu_{j,l} b_l,$$

and let

$$z_k = \sum_{j=1}^{N} D' c \eta_{k,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{k,j} \nu_{j,l} b_l.$$

We shall show that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq S$ and that the sets $\langle c z_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k}$ form a complete k-partite subgraph of G, completing the proof that part (b) of $\varphi(k, m, p, c, t)$ holds.

We show first that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq (b_0, b_1, \ldots, b_T)_{Q'', D''}$. So let $x \in (a_0, a_1, \ldots, a_t)_{p,c}$, and pick $\beta \in \{0, 1, \ldots, t\}$ and $\mu_{\beta+1}, \mu_{\beta+2}, \ldots, \mu_t \in [-p, p]$ such that

$$x = c a_\beta + \sum_{s=\beta+1}^{t} \mu_s a_s.$$

Then

$$x = \sum_{l \in M_\beta} c^2 (D')^2 b_l + \sum_{s=\beta+1}^{t} \sum_{l \in M_s} \mu_s c (D')^2 b_l.$$

Since $\min M_\beta < \min \bigcup_{s=\beta+1}^{t} M_s$, we have that the leading coefficient in this expansion is D'', while all other coefficients are at most $pc(D')^2 < Q''$.

Next we show that for each $i \in \{1, 2, \ldots, k-1\}$,

$$c z_i + [a_0, a_1, \ldots, a_t]_p \subseteq D'' z_i' + [b_0, b_1, \ldots, b_T]_{Q''},$$

and consequently the sets $\langle c z_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k-1}$ form a complete $(k-1)$-partite graph. To this end, let $\mu_0, \mu_1, \ldots, \mu_t \in [-p, p]$. Then

$$c z_i + \sum_{s=0}^{t} \mu_s a_s =$$

$$(D')^2 c^2 z_i' + (D')^2 c^2 b_0 + \sum_{l \in F} c D \nu_{0,l} b_l + \sum_{j=1}^{N} D' c^2 \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c \eta_{i,j} \nu_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \mu_s (D')^2 c b_l.$$

The coefficient on z_i' in this expansion is D'' while the coefficients on the b_l's have absolute value at most $2pc(Q')^2 < Q''$.

Finally we let $i \in \{1, 2, \ldots, k-1\}$, let $w \in c z_i + [a_0, a_1, \ldots, a_t]_p$, let $x \in c z_k + [a_0, a_1, \ldots, a_t]_p$, and show that there is an edge between w and x. Pick $\alpha_0, \alpha_1, \ldots, \alpha_s, \delta_0, \delta_1, \ldots, \delta_s \in [-p, p]$ such that $w = c z_i + \sum_{s=0}^{t} \alpha_s a_s$ and $x = c z_k + \sum_{s=0}^{t} \delta_s a_s$. Then

$$w = (D')^2 c^2 z_i' + (D')^2 c^2 b_0 + \sum_{l \in F} c D' \nu_{0,l} b_l + \sum_{j=1}^{N} D' c^2 \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c \eta_{i,j} \nu_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \alpha_s (D')^2 c b_l,$$

and

$$x = \sum_{j=1}^{N} D'c^2\eta_{k,j}b_j + \sum_{l\in F}\sum_{j=1}^{N} c\eta_{k,j}\nu_{j,l}b_l + \sum_{s=0}^{t}\sum_{l\in M_s} \delta_s(D')^2cb_l.$$

For $l \in F$ and $j \in \{0,1,\ldots,N\}$, let $\lambda_{j,l} = \nu_{j,l}$. For $s \in \{0,1,\ldots,t\}$ and $l \in M_s$, let $\lambda_{r,l} = D'\delta_s$, let $\lambda_{0,l} = D'\alpha_s - \eta_{i,r}\delta_s$, and for $j \in \{1,2,\ldots,N\} \setminus \{r\}$, let $\lambda_{j,l} = 0$. Note that each $|\lambda_{j,l}| \leq 2pQ'$. Note also that for $s \in \{0,1,\ldots,t\}$ and $l \in M_s$,

$$\sum_{j=1}^{N} \eta_{k,j}\lambda_{j,l} = (D')^2\delta_s$$

and

$$D'c\lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j}\lambda_{j,l}c = (D')^2c\alpha_s.$$

Then

$$\overline{\lambda} = ((\lambda_{0,N+1},\ldots,\lambda_{N,N+1}),(\lambda_{0,N+2},\ldots,\lambda_{N,N+2}),\ldots,(\lambda_{0,T},\ldots,\lambda_{N,T})) \in ([-2pQ',2pQ']^{N+1})^{T-N}$$

satisfies

(a) for each $l \in F$, $(\lambda_{0,l},\lambda_{1,l},\ldots,\lambda_{N,l}) = (\nu_{0,l},\nu_{1,l},\ldots,\nu_{N,l})$ and

(b) for each $s \in \{0,1,\ldots,t\}$ and each $l,v \in M_s$, $(\lambda_{0,l},\lambda_{1,l},\ldots,\lambda_{N,l}) = (\lambda_{0,v},\lambda_{1,v},\ldots,\lambda_{N,v})$.

Consequently, there is an edge between $\sum_{j=1}^{N}\eta_{k,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$ and

$$c^2(D')^2z_i' + c^2(D')^2b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N}\eta_{i,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l).$$

Now

$$\begin{aligned}
&\sum_{j=1}^{N}\eta_{k,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)\\
&\quad= \sum_{j=1}^{N}\eta_{k,j}c^2D'b_j + \sum_{l\in F}\sum_{j=1}^{N}\eta_{k,j}\nu_{j,l}cb_l + \sum_{s=0}^{t}\sum_{l\in M_s}\sum_{j=1}^{N}\eta_{k,j}\lambda_{j,l}cb_l\\
&\quad= \sum_{j=1}^{N}\eta_{k,j}c^2D'b_j + \sum_{l\in F}\sum_{j=1}^{N}\eta_{k,j}\nu_{j,l}cb_l + \sum_{s=0}^{t}\sum_{l\in M_s} c(D')^2\delta_sb_l\\
&\quad= x.
\end{aligned}$$

Also,

$$c^2(D')^2z_i' + c^2(D')^2b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N}\eta_{i,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$$

$$
\begin{aligned}
&= c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{j=1}^{N} \eta_{i,j} c^2 D' b_j + \sum_{l\in F}(D' c\lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j}\lambda_{j,l} c) b_l \\
&\qquad + \sum_{s=0}^{t}\sum_{l\in M_s}(D' c\lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j}\lambda_{j,l} c) b_l \\
&= c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{j=1}^{N} \eta_{i,j} c^2 D' b_j + \sum_{l\in F}(D' c\nu_{0,l} + \sum_{j=1}^{N} \eta_{i,j}\nu_{j,l} c) b_l + \sum_{s=0}^{t}\sum_{l\in M_s}(D')^2 c\alpha_s b_l \\
&= w.
\end{aligned}
$$

□

5 Independent arithmetic progressions, revisited

The following (in a more general form) was originally proved in [9] by application of the Hales-Jewett theorem. To illustrate a different approach in a special case, we now give a different proof, this time using Szemerédi's density theorem for arithmetic progressions [18] (which says that for any m and $\epsilon > 0$ there exists an n so that any set of ϵn elements from $[1, n]$ contains an m-term arithmetic progression).

Theorem 5.1 ([9]) *Fix k and ℓ. If G is a K_k-free graph on $\mathbb{N}$, then there exists an ℓ-term arithmetic progression which spans an independent set in G.*

Proof: Denote by $S(k, \ell)$ the following statement: There exists an integer $n = n(k, \ell)$ such that for every K_k-free graph G whose vertex set is an arithmetic progression of length n, there exists an arithmetic progression of length ℓ which is an independent set in G.

If for every k and ℓ, $S(k, \ell)$ holds, then Theorem 5.1 follows. For each fixed ℓ we will prove $S(k, \ell)$ by induction on k.

Observe that $S(2, \ell)$ is trivially true with $n(2, \ell) = \ell$. Suppose, therefore, that $S(k-1, \ell)$ holds and set $n^* = n(k-1, \ell)$. Let $n = n(k, \ell)$ be very large and consider a K_k-free graph G with vertex set $\{a, a+d, \ldots, a+(n-1)d\}$. Assume that G contains no independent set which is an arithmetic progression of length ℓ. Also, observe that $V(G)$ contains

$$(n-(\ell-1)) + (n-2(\ell-1)) + \ldots + \left(n - \left\lfloor \frac{n-1}{\ell-1} \right\rfloor (\ell-1)\right) \geq \frac{n^2}{3\ell}$$

arithmetic progressions of length ℓ. Since each of these arithmetic progressions contains an edge and each edge is contained in at most $\binom{\ell}{2}$ arithmetic progressions of length ℓ the graph G contains at least

$$\frac{n^2}{3\ell}\binom{\ell}{2}^{-1} \geq \frac{2n^2}{3\ell^3}$$

edges. This means that there exists a vertex x joined to at least $\frac{4n}{3\ell^3}$ other vertices. Now if n is sufficiently large compared to n^*, we may infer by Szemerédi's theorem that the neighborhood of x contains an arithmetic progression Y of length n^*. Since $\{x, y\} \in E(G)$ for every $y \in Y$ the subgraph $G[Y]$ of G induced by Y does not contain K_{k-1}. Thus, by the induction assumption

$S(k-1,\ell)$, the set Y contains an arithmetic progression of length ℓ which is an independent set in $G[Y]$ and hence also in G. □

We note that since an early draft of this paper, J. Solymosi (personal communication) has independently observed a similar proof of Theorem 5.1.

6 Concluding remarks

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