Finite Induced Graph Ramsey Theory: On Partitions of Subgraphs

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Abstract

For given finite (unordered) graphs G and H, we examine the existence of a Ramsey graph F for which the strong Ramsey arrow $F \longrightarrow (G)_r^H$ holds. We concentrate on the situation when H is not a complete graph. The set of graphs G for which there exists an F satisfying $F \longrightarrow (G)_2^{P_2}$ (P_2 is a path on 3 vertices) is found to be the union of the set of chordal comparability graphs together with the set of *convex* graphs.
KEYWORDS: Chordal, comparability, convex, graph, induced graph Ramsey theory.

1 Notation

For a set S and a given $n \in \omega$ we define $[S]^n = \{T \subseteq S : |T| = n\}$ to be the set of all subsets of S of size n. The power set of S is denoted by $\mathcal{P}(S)$.

For this discussion, a *hypergraph* $G = (V(G), E(G))$ is a finite vertex set $V(G)$ together with eges $E(G) \subseteq \mathcal{P}(V(G))$; for an (ordinary) *graph*, $E(G) \subseteq [V(G)]^2$. If H is a *weak subhypergraph* of G, i.e. $V(H) \subset V(G)$ and $E(H) \subseteq \mathcal{P}(V(H)) \cap E(G)$, we write $H \subseteq G$. If $H \subseteq G$ and $E(H) = \mathcal{P}(V(H)) \cap E(G)$ then we say H is an *induced* subhypergraph of G, denoted by $H \preceq G$. Letting $\cong$ denote graph isomorphism, we use the binomial coefficient $\binom{G}{H} = \{H' \preceq G : H' \cong H\}$.

An *ordered hypergraph* $(G, \leq)$ is a hypergraph G together with a total order $\leq$ on $V(G)$. Two ordered hypergraphs are isomorphic just in case there is an order preserving graph isomorphism between them. Definitions analogous to those given above hold for ordered hypergraphs as well. For a hypergraph H, let $\mathrm{ORD}(H)$ be the set of (distinct) isomorphism types of orderings of H. It is often convenient to abuse the notation and deliberately confuse an isomorphism type with a hypergraph of that given type and hence we write $\mathrm{ORD}(H) = \{(H, \leq_0), (H, \leq_1), \ldots, (H, \leq_{k-1})\}$.

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For a given (unordered) hypergraph H and an ordered hypergraph $(G, \leq^*)$ we define

$$\mathrm{DO}(H, G, \leq^*) = \left\{ (H, \leq) \in \mathrm{ORD}(H) : \binom{G, \leq^*}{H, \leq} \neq \emptyset \right\},$$

the distinct orderings of H in $(G, \leq^*)$. Let the minimum number of distinct orderings of H in any one ordered G be denoted by

$$\mathrm{mdo}(H, G) = \min\{|\mathrm{DO}(H, G, \leq)| : (G, \leq) \in \mathrm{ORD}(G)\}.$$

For example, if an ordinary graph H is complete, then $\mathrm{mdo}(H, G) \leq 1$ for any choice of G.

For hypergraphs F, G and H, and a fixed $r \in \omega$, we use the standard (strong) Ramsey arrow notation $F \longrightarrow (G)^H_r$ to mean that for any coloring $\Delta : \binom{F}{H} \longrightarrow r$, there exists $G' \in \binom{F}{G}$ so that Δ is constant on $\binom{G'}{H}$. We use the analogous notation for ordered graphs. The notation $\mathcal{R}[(G)^H_r] = \{F : F \longrightarrow (G)^H_r\}$ is used to denote the *Ramsey class* for G in coloring of H's with r colors. Observe that for these Ramsey type statements to be non-trivial we usually only consider pairs G, H so that $\mathrm{mdo}(H, G) \geq 1$. In ordinary graphs, we use P_n to refer to a path of length n on $n + 1$ vertices and $S_n = K_{1,n}$ for the star on $n + 1$ vertices.

2 Preliminaries

We recall *the* Ramsey theorem for ordered hypergraphs [1], [8], [9].

Theorem 2.1 *Given $r \in \omega$ and ordered hypergraphs $(G, \leq)$ and $(H, \leq)$, $\mathcal{R}[(G, \leq)^{(H,\leq)}_r] \neq \emptyset$.*

An application which will be used repeatedly in the remainder has appeared in [5]. For the purpose of exposition, we review the result here. Let $K = (X, \mathcal{K})$ be a hypergraph and recall that the *chromatic number* $\chi(K)$ of K is the least $n \in \omega$ so that there is an n-coloring of the vertex set X yielding no monochromatic edge $E \in \mathcal{K}$. If there is no such integer, we write $\chi(K) = \infty$. For a given pair of hypergraphs G and H, let us define a new hypergraph $K_{H,G}$ on the vertex set $\mathrm{ORD}(H)$ with edge set $E(K_{H,G}) = \{\mathrm{DO}(H, G, \leq^j) : (G, \leq^j) \in \mathrm{ORD}(G)\}$. Since for each ordering of G there corresponds an edge we may, by abuse of notation, refer to the orderings of G as edges, i.e., we could say $E(K_{H,G}) = \mathrm{ORD}(G)$, and a vertex $(H, \leq_i)$ is contained in an edge $(G, \leq^j)$ if and only if $(H, \leq_i) \preceq (G, \leq^j)$. We now give a characterization [5] of those triples H, G and r for which there exists a Ramsey graph.

Theorem 2.2 *Let G and H be hypergraphs. Then $\mathcal{R}[(G)^H_r] \neq \emptyset$ if and only if $\chi(K_{H,G}) > r$.*

Proof: Throughout the proof we fix $r \in \omega$, hypergraphs G, H and $K = K_{G,H}$.

Assume $\chi(K) > r$. Enumerate both $\mathrm{ORD}(H) = \{(H, \leq_0), (H, \leq_1), \ldots, (H, \leq_{t-1})\}$ and $\mathrm{ORD}(G) = \{(G, \leq^0), (G, \leq^1), \ldots, (G, \leq^{s-1})\}$. Construct a graph $(B, \leq) = \dot\bigcup_{j \in s}(G, \leq^j)$, the (disjoint) ordered sum of the orderings of G. By Theorem 2.1 choose $(B_0, \leq)$ satisfying

$$(B_0, \leq) \longrightarrow (B, \leq)^{(H,\leq_0)}_r,$$

and for $i = 1, \ldots, t-1$ choose (again by Theorem 2.1) successively $(B_i, \leq)$ so that

$$(B_i, \leq) \longrightarrow (B_{i-1}, \leq)_r^{(H, \leq_i)}.$$

We claim that B_{t-1}, the unordered version of $(B_{t-1}, \leq)$, satisfies $B_{t-1} \longrightarrow (G)_r^H$. Fix a coloring $\Delta : \binom{B_{t-1}}{H} \longrightarrow r$. By construction there exists $(B', \leq) \in \binom{B_{t-1}, \leq}{B, \leq}$ so that for each i, $\binom{B', \leq}{H, \leq_i}$ is monochromatic. This coloring of ordered H's in $(B', \leq)$ induces an r-coloring ψ of the vertices of K and since $\chi(K) > r$, there exists a $(G, \leq^j)$ in the edge set of K which is monochromatic with respect to ψ. Thus, there exists $G^* \in \binom{B_{t-1}}{G}$ monochromatic with respect to Δ, giving $B_{t-1} \in \mathcal{R}[(G)_r^H]$.

Now assume $\chi(K) \leq r$. Choose a coloring $\psi : \mathrm{ORD}(H) \longrightarrow r$ so that each element in $\mathrm{ORD}(G)$ is multi-colored. Examine any hypergraph F and impose an arbitrary (but fixed) ordering $\leq^*$ on $V(F)$. This naturally imposes an order on each $H' \in \binom{F}{H}$, so color $\binom{F}{H}$ according to ψ. That is, define $\Delta : \binom{F}{H} \longrightarrow r$ by $\Delta(H') = \psi((H', \leq^*))$ for each $H' \in \binom{F}{H}$, where $(H', \leq^*) \in \mathrm{ORD}(H)$ is the $\leq^*$-ordered H-subhypergraph. Since each element in $\mathrm{ORD}(G)$ is multi-colored with respect to ψ, so also is each $G' \in \binom{F}{G}$ with respect to Δ. This shows that $F \notin \mathcal{R}[(G)_r^H]$, and since F was arbitrary, $\mathcal{R}[(G)_r^H]$ is empty. $\Box$

This next result [5] can viewed either as a corollary to Theorem 2.1 or to 2.2.

Corollary 2.3 *Fix $r \in \omega$. If H and G are (unordered) hypergraphs satisfying $mdo(H, G) = 1$ then $\mathcal{R}[(G)_r^H] \neq \emptyset$.*

Proof: Let $\mathrm{mdo}(H, G) = 1$ and fix an ordering $\leq$ of G so that every induced H-subgraph of G is $\leq$-order-isomorphic to say $(H, \leq)$. Apply Theorem 2.1 to obtain $(F, \leq) \in \mathcal{R}[(G, \leq)_r^{(H, \leq)}]$. Using the condition $|\mathrm{DO}(H, G, \leq)| = 1$, it is now easy to check that the unordered F also satisfies $F \longrightarrow (G)_r^H$.

Alternatively, if $\mathrm{mdo}(H, G) = 1$, then $K_{H,G}$ contains a loop, so $\chi(K_{H,G}) = \infty$. $\Box$

Lemma 2.4 *Fix $r \in \omega$ and graphs B, H so that $\mathcal{R}[(B)_r^H] \neq \emptyset$. Then for all induced subgraphs $A \preceq B$, $\mathcal{R}[(A)_r^H] \neq \emptyset$.*

Proof: If $F \longrightarrow (B)_r^H$, then clearly $F \longrightarrow (A)_r^H$. $\Box$

3 Applications

An ordinary graph containing no cycles is a *forest*, and a connected forest is a *tree*.

Theorem 3.1 *If G is a forest, then $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$.*

Proof: If $\binom{G}{P_2} = \emptyset$ then the result is trivial. If $P_3 \not\preceq G$, then every connected component of G is a star. Clearly then $\mathrm{mdo}(P_2, G) = 1$ and the result follows by Corollary 2.3. So assume $P_3 \preceq G$. We will produce three orderings of G, namely $(G, \leq^0)$, $(G, \leq^1)$, and $(G, \leq^2)$, so that each of $\mathrm{DO}(P_2, G, \leq^i)$, $i \in 3$, is a unique pair from $\mathrm{ORD}(P_2)$. We then conclude that

$\chi(K_{P_2,G}) > 2$, (for we will have shown $K_{P_2,G}$ contains a triangle) and so by Theorem 2.2 the result will follow.

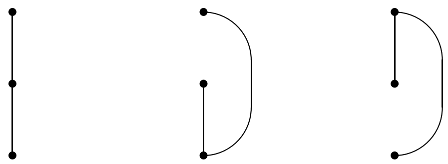
Fix a representation of G as a collection of rooted trees with at least one of these roots being an inner vertex of some copy of $P_3 \preceq G$. Let $V(G) = L_1 \dot\cup L_2 \dot\cup \cdots \dot\cup L_n$ be a partition of $V(G)$ into 'levels', that is, each L_j is the union of the j-th levels of all the rooted trees comprising G, where L_1 is the set of all the roots. Note that we have insisted that a copy of P_3 begins in L_2, goes 'down' to L_1, then back 'up' through L_2 and L_3. Impose an order $\leq^2$ on $V(G)$ which respects

$$L_1 \leq^2 L_2 \leq^2 L_3 \leq^2 \cdots \leq^2 L_n,$$

and let $\leq^1$ be the inverse order of $\leq^2$. Lastly, fix an order $\leq^0$ of $V(G)$ which 'folds' at levels, i.e.,

$$\cdots \leq^0 L_5 \leq^0 L_3 \leq^0 L_1 \leq^0 L_2 \leq^0 L_4 \leq^0 L_6 \leq^0 \cdots,$$

continuing until all levels are exhausted. Let $\mathrm{ORD}(P_2)$ be enumerated as in Figure 1.



$(P_2, \leq_0)$ $(P_2, \leq_1)$ $(P_2, \leq_2)$

Figure 1: $\mathrm{ORD}(P_2)$

It is straightforward to verify that for $i \in 3$, $\mathrm{DO}(P_2, G, \leq^i) = \{(P_2, \leq_j) : j \neq i\}$ as required. □

Throughout the remainder of this paper, we use the notation for the three orderings of P_2 as given in Figure 1.

Notice that we can not conclude from this proof that the resulting Ramsey graph is also a forest, even if it is minimal in some sense. Indeed, if $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$, G need not be a forest. If G is a triangle (a K_3), we trivially have $\mathcal{R}[(K_3)_2^{P_2}] \neq \emptyset$, just choose $F = G = K_3$. Furthermore, the orderings of the two graphs G_1 and G_2 in Figure 2 show $\mathrm{mdo}(P_2, G_i) = 1$ for $i = 1, 2$ and hence each $\mathcal{R}[(G_i)_2^{P_2}]$ is non-empty. Note that G_1 consists of n copies of K_3 attached at a single vertex, while G_2 is n copies of K_3 all sharing a common edge. Alternatively, we could say G_1 was constructed by starting with a star S_n, replacing each end-vertex with a copy of K_2 (edge) and then joining vertices of each K_2 in the same manner as the original vertex was. Similarly, G_2 could be conceived by replacing the central vertex of S_n with an edge in a like manner. As we have already observed, $\mathrm{mdo}(S_n, P_2) = 1$ and so $\mathcal{R}[(S_n)_2^{P_2}] \neq \emptyset$ also holds.

This method of replacing a vertex by a K_2 works in general. We first give a definition which generalizes that for a lexicographic product. Let G be a graph with a fixed enumeration $x_0, x_1, \ldots, x_{k-1}$ of $V(G)$. Let $K_{n_0}, K_{n_1}, \ldots, K_{n_{k-1}}$ be (vertex disjoint) complete (or null) graphs and define the product $G \otimes (n_0, n_1, \ldots, n_{k-1})$ on the vertex set $\dot\cup_{i \in k} V(K_{n_i})$ by

$$E(G \otimes (n_0, \ldots, n_{k-1})) = \dot\cup_{i \in k} E(K_{n_i}) \cup \{(y_i, y_j) : y_i \in V(K_{n_i}), y_j \in V(K_{n_j}), (x_i, x_j) \in E(G)\}.$$

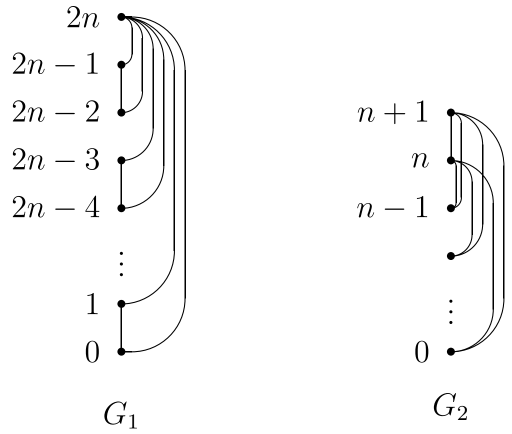


Figure 2: Vertices of a star exploded into edges.

In this product, we replace each vertex by a complete graph (possibly null) and connect each vertex of a 'replacement' graph to each vertex of another 'replacement' graph if and only if the replaced vertices were originally connected. If we let K_0 denote a null structure (a 'graph' with no vertices), and K_1 a single vertex, the graph $G \otimes (0, 1, 1, \ldots, 1) = G \setminus \{x_0\}$. For example, $K_4 \otimes (0, 1, 1, 1) = K_3$ and $K_3 \otimes (0, 1, 2) = K_3$.

In applying the definition of this product, we tacitly fix an enumeration of $V(G)$; our arguments do not depend on which. We remark that if $G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$, then $G \otimes (n_0+1, n_1, \ldots, n_{k-1}) = G' \otimes (2, 1, 1, 1, \ldots, 1)$ for some appropriate enumeration of $V(G')$. Using this type of inductive step, it is not hard to prove the following lemma:

Lemma 3.2 *If for each $i \in k$, $n_i, m_i \in \omega$ are given with each $n_i \leq m_i$, then*

$$G \otimes (n_0, n_1, \ldots, n_{k-1}) \preceq G \otimes (m_0, m_1, \ldots, m_{k-1})$$

The next theorem can be used to generate a large class of graphs G for which $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$ (for example, those obtained from forests by 'exploding' vertices).

Theorem 3.3 *Let $r \in \omega$, graphs G and H satisfy $\mathcal{R}[(G)_r^H] \neq \emptyset$ with $|V(G)| = k$. If for each edge (a, b) of H there exists $w \in V(H) \setminus \{a, b\}$ so that exactly one of $(w, a) \in E(H)$ or $(w, b) \in E(H)$ holds, then for any collection $n_0, n_1, \ldots, n_{k-1}$ of non-negative integers, $\mathcal{R}[(G \otimes (n_0, n_1, \ldots, n_{k-1}))_r^H] \neq \emptyset$ also holds.*

Proof: We first show the result for the case when each $n_i > 0$. In this case we use induction on $\sum_{i \in k} n_i$, the size of the vertex set of the product graph. The base step $n_0 = n_1 = \ldots = n_{k-1} = 1$ is the assumption. Fix positive integers n_0, n_1, $\ldots$, n_{k-1}; set $G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$ and $G'' = G \otimes (n_0+1, n_1, \ldots, n_{k-1})$, and assume $\mathcal{R}[(G')_r^H] \neq \emptyset$. It will suffice to show that $\mathcal{R}[(G'')_r^H] \neq \emptyset$.

For each ordering $\leq'$ of G' we will produce an ordering $\leq''$ of G'' so that $\mathrm{DO}(H, G', \leq')$ $= \mathrm{DO}(H, G'', \leq'')$. In this case, $K_{H,G'}$ will be a weak subhypergraph of $K_{H,G''}$ and so $\chi(K_{H,G''}) \geq \chi(K_{H,G'}) > r$ will give the result by Theorem 2.2.

Fix an ordering $\leq'$ of G'. Since $G'' = G' \otimes (2, 1, \ldots, 1)$, we can, without loss, take $V(G'') = V(G') \cup \{y\}$, where, say $x \in V(G')$ is replaced by x and y in G''. Define an ordering $\leq^{''}$ of G'', an extension of $\leq^{'}$, by keeping x and y adjacent in $\leq^{''}$ (and in the same relative position as was x in $\leq^{'}$). By Lemma 3.2, $\mathrm{DO}(H, G', \leq^{'}) \subseteq \mathrm{DO}(H, G'', \leq^{''})$, and so it remains to show the reverse inclusion.

Pick $H^* \in \binom{G''}{H}$ and let $(H^*, \leq) \preceq (G'', \leq^{''})$ be with the order induced by $\leq^{''}$. If $H^* \preceq G'$ there is nothing to show, so assume $y \in V(H)$, that is, H^* is a 'new' copy of H in G'' not in G'. If $x \notin V(H^*)$, then $(H*, \leq)$ is isomorphic to a copy of an ordered H already in G', namely the one with the vertex x replacing y in H^*. But if $x \in V(H^*)$ then $(x, y) \in E(H^*)$, and by the definition of the product, all remaining vertices of H^* are related to both x and y in identical manner, contrary to the condition in the statement of the theorem. So $(H^*, \leq)$ is of the same order-type as a copy of H already present in $(G', \leq^{'})$. Thus $\mathrm{DO}(H, G'', \leq^{''}) \subseteq \mathrm{DO}(H, G', \leq^{'})$, showing $\mathrm{DO}(H, G', \leq^{'}) = \mathrm{DO}(H, G'', \leq^{''})$ as required.

Now suppose some of the n_i's are zero. For each i, define $m_i = n_i$ if $n_i \neq 0$ and $m_i = 1$ if $n_i = 0$. Set $G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$ and $G'' = G \otimes (m_0, m_1, \ldots, m_{k-1})$. By the first case, $\mathcal{R}[(G'')^H_r] \neq \emptyset$, and by Lemma 3.2, $G' \preceq G''$, and so Lemma 2.4 gives the result. $\square$

Corollary 3.4 *Fix $r \in \omega$ and a connected triangle-free graph H with $|V(H)| \geq 2$. Let G be so that $\mathcal{R}[(G)^H_r] \neq \emptyset$ and $G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$ is defined. Then $\mathcal{R}[(G')^H_r] \neq \emptyset$.*

In particular, the above result holds for $H = P_2$.

4 Chordal, Comparability, and Convex Graphs

An ordinary graph is *chordal* (also called *triangulated* or a *rigid circuit*) if every cycle of length ≥ 4 has a chord, i.e., a chordal graph is a graph which contains no cycle on ≥ 4 vertices as an *induced* subgraph.

Lemma 4.1 *If a graph G is so that $\mathcal{R}[(G)^{P_2}_2] \neq \emptyset$ then G is chordal.*

Proof: Assume G is not chordal, i.e., there exists an induced cycle of length ≥ 4 in G. Then any ordering of G produces two distinct ordered P_2's as induced subgraphs, namely $(P_2, \leq_1)$ and $(P_2, \leq_2)$ (the ones which have the middle vertex at either end of the order).

Fix any graph F and impose an order $\leq$ on $V(F)$. Let $\Delta : \binom{F}{P_2} \longrightarrow 2$ be a coloring which satisfies

$$\Delta(P_2^{'}) = 0 \ \text{ if } \ (P_2^{'}, \leq) \cong (P_2, \leq_1),$$

and

$$\Delta(P_2^{'}) = 1 \ \text{ if } \ (P_2^{'}, \leq) \cong (P_2, \leq_2),$$

where $(P_2^{'}, \leq)$ is a copy of $P_2 \preceq F$ with the order $\leq$ imposed. Thus every $G' \in \binom{F}{G}$ is multicolored and so $F \notin \mathcal{R}[(G)^{P_2}_2]$, so if G is not chordal, then $\mathcal{R}[(G)^{P_2}_2] \neq \emptyset$. $\square$

A vertex x in an ordinary graph G is *simplicial* if its neighbors induce a complete subgraph of G. We use the following result of Dirac [2] (also see [3]).

Theorem 4.2 *Every chordal graph contains a simplicial vertex, and upon removal, produces another chordal graph.*

Given a partially ordered set $(Q, \leq)$, construct the graph $G(Q)$ on vertex set Q, where $(x, y) \in E(G)$ if and only if $x < y$ or $y < x$. Such a graph $G(Q)$ is called the *comparability graph for* $(Q, \leq)$. For a survey on comparability graphs, see [6].

Given a partial order $(Q, \leq)$, $(Q, \leq^*)$ is a *linear extension* of $(Q, \leq)$ if $\leq^*$ is a linear (total) order and $a \leq b$ implies $a \leq^* b$. Such a linear extension always exists.

An interesting (probably well known) characterization of comparibility graphs is the following. We remind the reader that $(P_2, \leq_0)$ is the 'flat' ordering of P_2 as in Figure 1.

Lemma 4.3 *G is a comparability graph if and only if G has an ordering $\leq^0$ so that $(P_2, \leq_0) \not\preceq (G, \leq^0)$.*

Proof: Let $G = G(Q)$ be a comparability graph for some poset $(Q, \leq)$. A linear extension $(Q, \leq^*)$ of $(Q, \leq)$ gives rise to the ordered graph $(G, \leq^*)$ in the following manner: for $x \leq^* y$, $(x, y) \in E(G, \leq^*)$ if and only if $x \leq y$. If (x, y) and (y, z) determine a weak $(P_2, \leq_0)$-subgraph of $(G, \leq^*)$, then transitivity of $\leq$ gives (x, z) to be an edge also, preventing an induced copy of $(P_2, \leq_0)$.

Now suppose that G has an ordering $\leq^0$ so that $(P_2, \leq_0) \not\preceq (G, \leq^0)$. Look at the relational structure $(Q, \leq)$ defined by $x \leq y$ if and only if $(x, y) \in E(G)$ and $x \leq^0 y$. If (x, y) and (y, z) are (ordered) edges of $(G, \leq^0)$, (x, z) is also, since $(G, \leq^0)$ does not contain a copy of $(P_2, \leq_0)$. Thus $x \leq z$ and the transitivity condition is satisfied for $(Q, \leq)$ to be a partial order and $G = G(Q)$ is a comparability graph. $\Box$

On the other hand, it also serves our purpose to classify those graphs having an ordering which admits only $(P_2, \leq_0)$, the 'flat' ordering of P_2.

Lemma 4.4 *Let $(G, \leq)$ be an ordering of G admitting only flat P_2's (i.e., $DO(P_2, G, \leq) \subseteq \{(P_2, \leq_0)\}$). Then there exists an order $\leq^*$ so that $(G, \leq^*)$ also admits only flat P_2's, and the connected components of G determine disjoint intervals in the order. Also, if $(x, y) \in E(G)$ with $x \leq^* y$, then the set of vertices $\{z : x \leq^* z \leq^* y\}$ induces a complete subgraph of G.*

Proof: Let $(G, \leq)$ admit only flat P_2's. It is easy to see that there is $\leq^*$ so that $(G, \leq^*)$ also admits only flat P_2's, and components of G determine disjoint intervals in the order $\leq^*$.

The proof we give for the last statement of the theorem is by induction on $|V(G)|$. By the first part, we can assume without loss that G is connected. For $|V(G)| \leq 3$ the result is trivial, so let $v_0 \leq^* v_1 \leq^* \ldots \leq^* v_n$ is an enumeration of $V(G)$ where $\mathrm{DO}(P_2, G, \leq^*) \subseteq \{(P_2, \leq_0)\}$. Let G' be the graph induced by $V(G) \backslash \{v_n\}$ and observe that since the deletion of a vertex can not create any new copies of P_2, $\mathrm{DO}(P_2, G', \leq^*) \subseteq \{(P_2, \leq_0)\}$. Thus, by the induction hypothesis, G' satisfies the lemma. Since G does not admit any copies of $(P_2, \leq_2)$, it follows that the graph induced by the neighbors of v_n is complete, and since G is assumed to be connected, so is all of G'. It now follows that for each $i \in n-2$, $(v_i, v_{i+1}) \in E(G)$. Let v_j be the least (in the order $\leq^*$) neighbor of v_n.

It is sufficient to show that $\{v_i : j \leq i \leq n\}$ is a clique. Recursively, the pairs (v_{j+1}, v_n), (v_{j+2}, v_n), …, (v_{n-1}, v_n) can be shown to be edges to avoid copies of $(P_2, \leq_2)$. Also, for each

k, l with $j \leq k < l \leq n-1$, similarly $(v_k, v_l) \in E(G)$ can be shown since (v_k, v_n) and (v_l, v_n) are edges and G forbids copies of $(P_2, \leq_1)$.. $\square$

Roughly speaking, we see that those graphs having an ordering which admits only flat P_2's can be constructed by fixing a collection of intervals in an ordered set of vertices and imposing a complete graph on vertices determined by each interval in the collection. In fact, the converse holds as well. Any ordered graph constructed in this manner can easily be seen to omit flat P_2's.

Recall that a subset S of a partial order $(P, \leq)$ is called *convex* if whenever $x, y \in S$ and $x \leq z \leq y$ then $z \in S$. So in this respect, ordered graphs satisfying Lemma 4.4 have the property that if a subset of vertices determines a clique, then it corresponds to an interval in the linear order and hence is convex. Hence, we call those graphs G for which there exists an ordering $\leq^*$ of G so that $\mathrm{DO}(P_2, G, \leq^*) \subseteq \{(P_2, \leq_0)\}$ *convex clique graphs*, or simply *convex*, without having to specify an ordering. This terminology avoids any conflation with the term 'interval graph', yet captures the property.

5 Complete Classification

We can now classify those graphs G for which $\mathcal{R}[(G)_2^{P_2}]$ is non-empty.

Theorem 5.1 $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$ *if and only if either G is a chordal comparability graph or G is convex.*

Proof: First assume that G is chordal and is a comparability graph. We define three orderings of G as follows.

By Theorem 4.2 there exists a simplicial vertex $s_0 \in V(G)$. By the same theorem, there is $s_1 \in V(G) \setminus \{s_0\}$, again simplicial. Continue, exhausting $V(G)$ and let $\leq^1$ be an ordering of $V(G)$ given by $s_0 \leq^1 s_1 \leq^1 \ldots \leq^1 s_{|V(G)|-1}$. Observe that $(P_2, \leq_1) \not\leq (G, \leq^1)$, because each upper (right) neighborhood of each vertex is complete. Similarly define $(G, \leq^2)$ where $\leq^2 = (\leq^1)^{-1}$. Then $(P_2, \leq_2) \not\leq (G, \leq^2)$. Now let $(G, \leq^0)$ be the ordered graph guaranteed by Lemma 4.3 so that $(P_2, \leq_0) \not\leq (G, \leq^0)$. So by Theorem 2.2, $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$.

Now assume that G is convex. If $\leq^*$ is an ordering of $V(G)$ so that $\mathrm{DO}(P_2, G, \leq^*) = \{(P_2, \leq_0)\}$, then by Theorem 2.2 (or Corollary 2.3) we have $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$ as well.

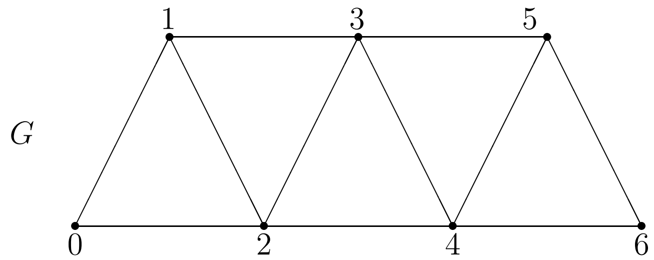
To prove the other direction, suppose that $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$. Then by Lemma 4.1, G must chordal. It remains to show that either G is a comparability graph or G is convex. We will use Theorem 2.2 and two orderings given by chordality in the first part of the proof.

As defined for Theorem 2.2, set $K = K_{P_2,G}$ on vertices $(P_2, \leq_0)$, $(P_2, \leq_1)$, and $(P_2, \leq_2)$. By chordality, fix two hyperedges of K, i.e., two orderings of G, each omitting $(P_2, \leq_1)$ and $(P_2, \leq_2)$ respectively. If either of these two orderings of G omits $(P_2, \leq_0)$ as well, (i.e., if either corresponds to a hyperedge of K consisting of a single vertex—a loop) we are done since then G is a comparability graph by Lemma 4.3. So suppose that both $\{(P_2, \leq_0), (P_2, \leq_1)\}$ and $\{(P_2, \leq_0), (P_2, \leq_2)\}$ are hyperedges of K, and neither $\{(P_2, \leq_1)\}$ or $\{(P_2, \leq_2)\}$ are hyperedges. Since $\chi(K) \geq 3$, either $\{(P_2, \leq_1), (P_2, \leq_2)\}$ or $\{(P_2, \leq_0)\}$ is a hyperedge of K. In the first case, the edge omits $(P_2, \leq_0)$ and so by Lemma 4.3, G is a comparability graph and we are done. In the second case, G is convex and we are done again. $\square$

We add that although a convex graph is chordal, it is not necessarily a comparability graph. For the purpose of presenting an example of such a convex graph, we recall the following well known characterization theorem [4] (see [6] or [7] for other references) for comparability graphs.

Theorem 5.2 *G is a comparability graph if and only if G does not contain an odd number of* (not necessarily distinct) *vertices $v_0, v_1, \ldots, v_n = v_0$, so that for each i, $(v_i, v_{i+1}) \in E(G)$, but $(v_i, v_{i+2}) \notin E(G)$.*

Example 5.3 *The graph G given by $V(G) = \{0, 1, \ldots, 6\}$ and $E(G) = \{(i, i+1) : i \in 6\} \cup \{(i, i+2) : i \in 5\}$ is convex but is not a comparability graph.*



Proof: The natural ordering of vertices as given is used to show that G is convex. The 'semicycle' $\langle 0\ 2\ 3\ 5\ 3\ 1\ 3\ 4\ 6\ 4\ 2 \rangle$ on 11 vertices satisfies the condition of Theorem 5.2. □

This appears to be the smallest of infinitely many such examples. It would be of interest to classify chordal comparability graphs. For examples of graphs which are one or the other and not both, see [7]. In classifying those graphs for which $\mathcal{R}[(G)_2^{P_2}]$ is non-empty, the case for colorings of the graph consisting of an edge and a disjoint vertex is also implicitly settled (examine the Ramsey statements in the complement).

References

[1] F. G. Abramson and L. A. Harrington, Models without indiscernibles, *J. Symbolic Logic* **43** (1978), 572–600.

[2] G. A. Dirac, On rigid circuit graphs. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **25** (1961), 71–76.

[3] D. R. Fulkerson and O. A. Gross, Incidence matrices and interval graphs, *Pacific J. Math.* **15** (1965), 835–855.

[4] A. Ghoila-Houri, Caracterisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graph d'une relation d'ordre, *C. R. Acad. Sci. Paris* **254** (1962), 1370–1371.

[5] D. S. Gunderson, V. Rödl, and N. W. Sauer, Some results in finite graph Ramsey theory, *Congressus Numerantium* **79** (1990), 232–243.

[6] D. Kelly, Comparability Graphs, in *Graphs and Order*, I. Rival ed., D. Reidel Publishing Company, Boston (1985), 3–40.

[7] R. H. Möhring, Algorithmic aspects of comparability graphs, in *Graphs and Order*, I. Rival ed., D. Reidel Publishing Company, Boston (1985), 41–101.

[8] J. Nešetřil and V. Rödl, Partitions of finite relational and set systems *J. Combin. Theory Ser. A* **22** (1977), 289–312.

[9] J. Nešetřil and V. Rödl, Ramsey classes of set systems, *J. Combin. Theory Ser. A* **34** (1983), 183–201.