Intersection statements for systems of sets

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Abstract

A family of r sets is called a Δ-*system* if any two sets have the same intersection. Denote by $F(n,r)$ the most number of subsets of an n-element set which do not contain a Δ-system consisting of r sets. Constructive new lower bounds for $F(n,r)$ are given which improve known probabilistic results, and a new upper bound is given by employing an argument due to Erdős and Szemerédi. Another construction is given which shows that for certain n, $F(n,3) \geq 1.551^{n-2}$. We also show a relationship between the upper bound for $F(n,3)$ and the Erdős–Rado conjecture on the largest uniform family of sets not containing a Δ-system.

1 Introduction

A family $\mathcal{F}$ of sets is called k-*uniform* if for every $F \in \mathcal{F}$, $|F| = k$ holds. A family of sets is called a Δ-*system* if any two sets have the same intersection. Define $f(k,r)$ to be the least integer so that any k-uniform family of $f(k,r)$ sets contains a Δ-system consisting of r sets. Erdős and Rado [8] proved that

$$(r-1)^k < f(k,r) < k!(r-1)^k \tag{1}$$

and conjectured that for each r, there exists a constant C_r so that $f(k,r) < C_r^k$. Erdős (see [6]) has offered 1000 dollars for the proof or disproof of this

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for $r = 3$. Several authors (Abbott, Hanson, and Sauer [3], Abbott and Hanson [4], Spencer [14], and Kostochka [12, 13]) have slightly improved the bounds in (1) but a proof or disproof of the conjecture is nowhere in sight. Currently, the best known upper bound [13] is

$$f(k,r) < Ck!\left(\frac{(\log\log\log k)^2}{\alpha\log\log k}\right)^k, \tag{2}$$

where α is any positive constant and k is large enough. As far as the lower bounds are concerned, limited progress seems to have been made since 1974 (see [1], [2], [4]). Infinite versions have also been studied in, for example, [7] and [9].

What appeals to us here is the similar problem for families having a fixed ground set. Define $F(n,r)$ to be the largest integer so that there exists a family $\mathcal{F}$ of subsets of an n-element set which does not contain a Δ-system of r sets. In [10], Erdős and Szemerédi showed

$$F(n,3) < 2^{n(1-\frac{1}{10\sqrt{n}})} \tag{3}$$

and stated that the probabilistic method implies that for each $r \geq 3$, there exists a constant $c_r > 0$, so that

$$F(n,r) > (1+c_r)^n$$

where $c_r \to 1$ as $r \to \infty$. Let

$$\beta_r = \lim_{n\to\infty} F(n,r)^{1/n}.$$

Abbott and Hanson [5] observed that β_r exists and that the probabilistic method mentioned above gives $\beta_r \geq 2(r+2)^{-1/r}$. They also presented a construction implying

$$\beta_r \geq \binom{2r-2}{r}^{1/(2r-2)} \sim\ 2^{(1-\frac{\log(2r)}{4r})}. \tag{4}$$

The Erdős-Szemerédi proof [10] of (3) reveals relations between bounds for $f(k,r)$ and $F(n,r)$. It shows that good upper bounds for $f(k,r)$ yield satisfactory upper bounds for $F(n,r)$ and strong lower bounds (if found) for $F(n,r)$ might imply lower bounds for $f(k,r)$. In Section 2, we repeat the Erdős-Szemerédi argument, however giving a more general outcome (Theorem 2.1) which yields the following two propositions.

Proposition 1.1 *For each r and sufficiently large n,*

$$F(n,r) < 2^{n - \frac{\sqrt{n \log\log n}}{\log\log\log n}}.$$

The second consequence of Theorem 2.1 is the next proposition showing that if the Erdős–Rado conjecture is true, then there exists an $\epsilon > 0$ so that for large n, $F(n,3) < (2-\epsilon)^n$.

Proposition 1.2 *If there exists a constant C so that $f(k,3) < C^k$, then for n sufficiently large,*

$$F(n,3) < 2^{n(1-0.65/C)}.$$

In particular, $\beta_r \leq 2^{(1-1/2C)}$.

A *weak* Δ-system is a family of sets where all pairs of sets have the same intersection size. Frankl and Rödl [11] proved that an upper bound of the form $(2-\epsilon)^n$ holds for the size of any family of subsets of an n element set not containing a weak Δ-system of 3 sets. This together with Proposition 1.2 motivates obtaining lower bounds on $F(n,r)$ and β_r. In Section 3 we give a bound for general r, improving (4).

Theorem 1.3 *For every $r \geq 3$ and every n of the form $n = 2pr\lfloor \log r \rfloor$,*

$$F(n,r) \geq 2^{n(1-\frac{\log\log r}{2r} - O(1/r))},$$

(and there are uniform families which witness this bound). In particular,

$$\beta_r \geq 2^{(1-\frac{\log\log r}{2r} - O(1/r))}.$$

In Section 4, we concentrate on $r = 3$ and derive the following.

Theorem 1.4 *For every n of the form $n = 14q$,*

$$F(n,3) \geq 1.53^n.$$

Refining the argument, we also obtain

Theorem 1.5 *For every n of the form $n = 48q + 2$,*

$$F(n,3) \geq 1.551^{n-2}.$$

In particular, $\beta_3 \geq 1.551$.

In our proofs, it will be convenient to use the shorthand *r-free family* of sets to denote a family which contains no Δ-system consisting of r sets.

2 Analyzing the Erdős-Szemerédi proof

Repeating the Erdős-Szemerédi argument, we show that it indeed proves more than was originally claimed.

Theorem 2.1 ([10]) *Let r be fixed. Suppose that for $k > k_0$, $\alpha = \alpha(k)$ satisfies $f(k,r) \leq \alpha^k$. For n sufficiently large, if $k > n^{0.1}$ and*

$$2k\alpha < 1.31n, \tag{5}$$

then

$$F(n,r) < 2^{n-k}.$$

Proof of Theorem 2.1: Let $\mathcal{A} = \{A_i \mid 1 \leq i \leq t\}$ be the largest r-free family of subsets of an n-element set S, and for each $l = 1, \ldots, n$, $\mathcal{A}_l$ be the subfamily of $\mathcal{A}$ with members of cardinality l. Obviously, there is an l so that $s = |\mathcal{A}_l| \geq t/n$. For each $A_i \in \mathcal{A}_l$, consider all its subsets of size $l-k$. The total number of such subsets is easily bounded from above by $s\binom{l}{k}$. The total number of subsets of S of size $l-k$ is clearly $\binom{n}{l-k}$, and so, some set B of size $l-k$ occurs in at least u members of $\mathcal{A}_l$, where

$$u \geq \frac{s\binom{l}{k}}{\binom{n}{l-k}} = \frac{s\binom{n+k}{k}}{\binom{n+k}{l}} > s\binom{n+k}{k}2^{-n-k}.$$

Let $\mathcal{A}_{l,B} = \{A_i \in \mathcal{A}_l \mid A_i \supset B\}$. Then $\mathcal{A}_{l,B} - B = \{A_i \setminus B \mid A_i \in \mathcal{A}_{l,B}\}$ is a k-uniform r-free family. Thus, $u < f(k,r)$ and so,

$$t \leq ns < n \cdot f(k,r)2^{n+k}\binom{n+k}{k}^{-1} < n^2 \cdot 2^n \left(\frac{2k\alpha}{ne}\right)^k.$$

By (5), the last expression does not exceed $n^2 \cdot 2^n(\frac{1.31}{e})^k < 2^{n-k}$. $\square$

This correlation between $f(k,r)$ and $F(n,r)$ enables easy proofs of Propositions 1.1 and 1.2.

Proof of Proposition 1.1: By (2), for large k,

$$f(k,r) < \left(\frac{k(\log\log\log k)^2}{10\log\log k}\right)^k.$$

Thus, for n sufficiently large and $k = \frac{\sqrt{n \cdot \log\log n}}{\log\log\log n}$, the conditions of Theorem 2.1 hold. Hence $F(n,r) < 2^{n-k}$. $\square$

Proof of Proposition 1.2: Let k, n be large, $f(k,3) < C^k$, then for $k = \lceil 0.65n/C \rceil$, (5) holds, and by Theorem 2.1, we get what was promised. $\square$

3 A lower bound for large r

Let $V_1, V_2, \ldots, V_p$ be pairwise disjoint finite sets and for each $i = 1, \ldots, p$, let $\mathcal{F}_i$ be a family of subsets on V_i. Define $\prod_{i=1}^{p} \mathcal{F}_i$ to be the family of subsets A of $\bigcup_{i=1}^{p} V_i$ such that $A \cap V_i \in \mathcal{F}_i$ holds for each $i = 1, \ldots, p$. Clearly,

$$|\prod_{i=1}^{p} \mathcal{F}_i| = \prod_{i=1}^{p} |\mathcal{F}_i|. \tag{6}$$

If all pairs $(V_i, \mathcal{F}_i)$ are copies of one pair $(V, \mathcal{F})$, we shall denote $\prod_{i=1}^{p} \mathcal{F}_i$ by $\mathcal{F}^p$. A family of sets is said to be *Sperner* (or "has the Sperner property") if none of the sets contains another one.

The following lemma is a relative of Theorem 1 in [1].

Lemma 3.1 *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are Sperner r-free families on disjoint ground sets V_1 and V_2 then $\prod_{i=1}^{2} \mathcal{F}_i$ is also a Sperner r-free family.*

Proof of Lemma 3.1: Let $A, B \in \prod_{i=1}^{2} \mathcal{F}_i$. For some $i \in \{1,2\}$, $A \cap V_i \neq B \cap V_i$. Then by the Sperner property of $\mathcal{F}_i$, both $(A \cap V_i) \setminus (B \cap V_i)$ and $(B \cap V_i) \setminus (A \cap V_i)$ are non-empty. It follows that $\prod_{i=1}^{2} \mathcal{F}_i$ is Sperner.

Suppose now that $A_1, \ldots, A_r \in \prod_{i=1}^{2} \mathcal{F}_i$ form a Δ-system of r sets. Let $i \in \{1,2\}$ be such that not all the sets $A'_j = A_j \cap V_i$ coincide. Without loss of generality, we assume that for $K = A'_1 \cap A'_2$, $K \neq A'_1$. By the Sperner property of $\mathcal{F}$ then $K \neq A'_2$. Since $A_1, \ldots, A_r$ form a Δ-system, $K \subset A'_j$, for each $j = 1, \ldots, r$ and no element in $V_i \setminus K$ belongs to more than one of the A'_j-s. It follows that all A'_j-s are distinct and form a Δ-system of r sets. This is a contradiction. $\square$

We use the notation $[n]^k = \{S \subseteq \{1, \ldots, n\} : |S| = k\}$. The next lemma is very similar to that in [5] (the consequence of which is mentioned in the introduction).

Lemma 3.2 *For any $k \geq r+2$, the family $[2r]^k$ is r-free.*

Proof of Lemma 3.2: Suppose that $A_1, \ldots, A_r \in [2r]^k$ form a Δ-system of r sets. Let m be the size of their common intersection M. Then all the sets $A_i \setminus M$ are disjoint and so counting the elements used in the Δ-system, we have

$$m + r(k-m) \geq m + r(r+2-m) \geq r(r+2) - (r-1)(r+1) = 2r+1,$$

which is impossible. $\square$

For $t, r \geq 1$, let $V_1, \ldots, V_t$ be pairwise disjoint sets of cardinality $2r$ and $W = \bigcup_{i=1}^t V_i$. Define $\mathcal{F}(r,t)$ to be the collection of all subsets A of W satisfying

$$|A \cap V_i| \in \{r+2, r+2+t, \ldots, r+2+t\lfloor (r-2)/t \rfloor\} \tag{7}$$

for each $i = 1, \ldots, t$.

Lemma 3.3 *For any r and t, the family $\mathcal{F}(r,t)$ is r-free and contains a uniform (and hence Sperner) subfamily $\mathcal{F}'(r,t)$ of cardinality at least $|\mathcal{F}(r,t)|/r$.*

Proof of Lemma 3.3: Suppose that $A_1, \ldots, A_r \in \mathcal{F}(r,t)$ form a Δ-system of r sets. For each $i = 1, \ldots, t$ and $j = 1, \ldots, r$ set $A_j(i) = A_j \cap V_i$.

Let $B(i) = A_1(i) \cap A_2(i)$. Since $A_1, \ldots, A_r$ form a Δ-system, $B(i) \subseteq A_j(i)$ for each j, and each element of $V_i \setminus B(i)$ belongs to at most one of the A_j-s. Like in the proof of Lemma 3.2, we observe that it is impossible to have all $A_j(i)$-s distinct from the corresponding $B(i)$, so let $A_{l(i)}(i) = B(i)$. By (7), each $A_j(i)$ is distinct from $A_{l(i)}(i)$ and has at least t elements in $A_j(i) \setminus A_{l(i)}(i)$ which should coincide with $A_j(i) \setminus \bigcup_{l \neq j} A_l(i)$. Hence the number of such sets is at most $(2r - (r+2))/t$. Consequently, for at least 2 members of $\{A_1, \ldots, A_r\}$, their intersections with V_i are equal to $B(i)$ for each i. This is a contradiction.

Observe that the size of any member of $\mathcal{F}(r,t)$ belongs to the set $\{t(r+2), t(r+3), \ldots t(r+r-2)\}$. It follows that for some i, the size of $\{A \in \mathcal{F}(r,t) : |A| = t(r+i)\}$ is at least $|\mathcal{F}(r,t)|/r$. $\square$

Proof of Theorem 1.3: Because of the $O(1/r)$ in the statement of Theorem 1.3, we may assume that r is large enough. Put $t = \lfloor \log_2 r \rfloor$, and let $n = p \cdot 2rt$.

Let $\mathcal{F}'(r,t)$ be the family provided by Lemma 3.3. By Lemma 3.1, the family $(\mathcal{F}'(r,t))^p$ does not contain any Δ-system of r sets. The number of subsets A of a V_i satisfying (7) is at least $(1 - O(1/\sqrt{r})) \cdot 2^{2r-1}/t$. Consequently, for large r,

$$|\mathcal{F}'(r,t)| \geq |\mathcal{F}(r,t)|/r \geq (2^{2r-1}(1 - O(1/\sqrt{r}))/t)^t/r \geq \frac{2^{2tr-t}}{t^t 2r} \geq \frac{2^{2tr}}{t^t 2r^2}.$$

Thus,

$$\begin{aligned} |(\mathcal{F}'(r,t))^p| &\geq \left[\frac{1}{2r^2}\left(\frac{2^{2r}}{t}\right)^t\right]^{\frac{n}{2rt}} \\ &= 2^{n(1-\frac{\log\log r}{2r} - O(1/r))} . \ \Box \end{aligned}$$

4 A lower bound for $r = 3$

4.1 Outline of the construction

To arrive at Theorem 1.5 we first present a Sperner 3-free family $\mathcal{F}$ comprised of subsets of a 14-element "brick". With $\mathcal{F}$ and Lemma 3.1 we then prove Theorem 1.4. On another 14-element brick we construct another Sperner 3-free family $\mathcal{L}$. We then give another product lemma, and apply it to combine $\mathcal{F}$ and $\mathcal{L}$, yielding a family $\mathcal{Q}$ on a ground set of 26 elements. Applying the product lemma again to two disjoint copies of $\mathcal{Q}$ produces a family $\mathcal{R}$ on a ground set of 50 vertices. Finally, we take the product of $\mathcal{R}$ with itself, producing $\mathcal{R}^2$ on 98 vertices, then by successively taking the product of the result with $\mathcal{R}$ again, each time increase the existing ground set by 48 until we reach n.

4.2 The family $\mathcal{F}$ on a 14 element brick

To begin the construction, let $W = \{w_1, \ldots, w_5, y\}$ and define four families $\mathcal{H}_0, \ldots, \mathcal{H}_3$ of subsets of W as follows. Put $\mathcal{H}_0 = \{\emptyset\}$ and $\mathcal{H}_1 = \{A \subset W : |A| = 5\}$. The family $\mathcal{H}_2$ will be the following family of triples of elements of W:

$$\mathcal{H}_2 = \bigcup_{i=1}^{5} \{\{y, w_i, w_{i+1}\}, \{w_i, w_{i-2}, w_{i+2}\}\},$$

where the indices are taken modulo 5. Finally, let $\mathcal{H}_3 = \{W \setminus A : A \in \mathcal{H}_2\}$. The following known fact (see [2], [3]) can be verified directly.

Lemma 4.1 *The family $\mathcal{H}_2$ is intersecting, Sperner, and 3-free. Moreover, $\mathcal{H}_3$ is isomorphic to $\mathcal{H}_2$.* $\square$

The ground set X for our desired family $\mathcal{F}$ consists of two copies W_1 and W_2 of W and two additional elements x_1 and x_2 (in total, $|X| = 14$). Subfamilies of $\mathcal{F}$ shall be described by quadruples of the type $\langle i_1, i_2, j_1, j_2 \rangle$, where i_1 and i_2 will take values from $\{0, 1, 2, 3\}$ and j_1, j_2 from $\{0, 1\}$. Now we are ready to indicate $\mathcal{F}$ on X. We define $\mathcal{F} = \bigcup_{t=1}^{8} \mathcal{F}_t$, where $\mathcal{F}_t = \langle i_1, i_2, j_1, j_2 \rangle$ consists of exactly those subsets A of X with the following property for $q = 1, 2$: $A \cap W_q \in \mathcal{H}_{i_q}$ and A contains exactly j_s elements of the set $\{x_s\}$, $s = 1, 2$. Let

$$
\begin{aligned}
\mathcal{F}_1 &= \langle 1, 1, 0, 0 \rangle, \\
\mathcal{F}_2 &= \langle 2, 2, 1, 1 \rangle, \\
\mathcal{F}_3 &= \langle 1, 0, 1, 1 \rangle, \\
\mathcal{F}_4 &= \langle 0, 1, 1, 1 \rangle, \\
\mathcal{F}_5 &= \langle 1, 2, 1, 0 \rangle, \\
\mathcal{F}_6 &= \langle 3, 1, 1, 0 \rangle, \\
\mathcal{F}_7 &= \langle 1, 3, 0, 1 \rangle, \\
\mathcal{F}_8 &= \langle 2, 1, 0, 1 \rangle.
\end{aligned}
$$

It will be of some help that for $t = 3, 5, 7$, $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$ are symmetric with respect to W_1 and W_2, and for $t = 5, 6$, $\mathcal{F}_t$ and $\mathcal{F}_{t+2}$ are symmetric with respect to x_1 and x_2.

Lemma 4.2 *The family $\mathcal{F}$ defined above is Sperner, 3-free, and satisfies $|\mathcal{F}| = 388$.*

Proof of Lemma 4.2: By definition, $|\mathcal{F}_1| = |\mathcal{H}_1|^2 = 36$, $|\mathcal{F}_2| = |\mathcal{H}_2|^2 = 100$, $|\mathcal{F}_3| = |\mathcal{F}_4| = 6$, $|\mathcal{F}_5| = \ldots = |\mathcal{F}_8| = 60$. Thus, $|\mathcal{F}| = 388$.

To derive the Sperner property, observe first that each member of $\mathcal{F}_t$ has cardinality k_t, where $k_1 = 10, k_2 = 8, k_3 = k_4 = 7, k_5 = \ldots = k_8 = 9$. Notice that only the members of $\mathcal{F}_1$ do not meet $\{x_1, x_2\}$ and hence none of them contains any other member of $\mathcal{F}$. The members of $\mathcal{F}_5 \cup \ldots \cup \mathcal{F}_8$ have smaller intersection size with $\{x_1, x_2\}$ than those of $\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. The members of $\mathcal{F}_2$ have smaller intersection size with W_1 than those of $\mathcal{F}_3$ and have smaller intersection size with W_2 than those of $\mathcal{F}_4$. Thus, $\mathcal{F}$ is Sperner.

Suppose that some members A, B and C of $\mathcal{F}$ form a Δ-system. We have to consider several cases. For $0 \le p, q \le 3$ we denote by case $[p, q]$ the case when x_1 belongs to exactly p many of A, B and C, and x_2 belongs to q of them. Since A, B and C form a Δ-system, the value 2 is forbidden for p and q. We also take into account the symmetry between p and q. In each case we shall find an element which belongs to exactly two of A, B and C, yielding a contradiction.

Case $[3, 3]$. Then A, B and C belong to $\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. By Lemmas 4.1 and 3.1, not all three of A, B, and C belong to $\mathcal{F}_2$. We may assume $A \in \mathcal{F}_3$. If another one, say B also belongs to $\mathcal{F}_3$, then no other member of $\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$ covers their intersection (of size 4) which is a contradiction. If both B and C belong to $\mathcal{F}_2$ then their common element in W_2 (which exists by Lemma 4.1) is what we are after. The last possibility is that $B \in \mathcal{F}_2$ and $C \in \mathcal{F}_4$. Then each element of $W_1 \cap A \cap B$ belongs to exactly two of the sets A, B and C.

Case $[3, 1]$. Then two of the sets A, B and C belong to $\mathcal{F}_5 \cup \mathcal{F}_6$. First assume that $A \in \mathcal{F}_5$, $B \in \mathcal{F}_5 \cup \mathcal{F}_6$ and $C \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. If $B \cap W_1 \neq C \cap W_1$, then the symmetric difference between $B \cap W_1$ and $C \cap W_1$ has size at least two, and hence meets $A \cap W_1$. This gives an element which belongs to A and moreover to exactly one of B and C. Secondly, suppose $B \cap W_1 = C \cap W_1$. Then $C \in \mathcal{F}_3$ and $B \in \mathcal{F}_5$. In this case, A and B have a common element in W_2 which does not intersect C. Thirdly, let both A and B be in $\mathcal{F}_6$. In order that C covers $A \cap B \cap W_2$, we need $C \in \mathcal{F}_4$. As in the second subcase a common element of A and B in W_2 does not intersect C.

Case $[3, 0]$. We may assume that A and B are in $\mathcal{F}_5$, and $C \in \mathcal{F}_5 \cup \mathcal{F}_6$. We can also assume that $|A \cap B \cap W_1| \ge |A \cap B \cap W_2|$. If not all of A, B and C coincide on W_1, then the intersection $|A \cap B \cap W_1|$ is not contained in C. So, let A, B and C coincide on W_1. Then their corresponding intersections with W_2 form a subfamily of $\mathcal{H}_3$, which contradicts Lemma 4.1.

Case $[1, 1]$. If two of A, B and C belong to $\mathcal{F}_1$, then the intersection of these two has at least eight elements in common with $W_1 \cup W_2$. But any member of $\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$ has at most six elements in $W_1 \cup W_2$. So, we may assume $A \in \mathcal{F}_1$, $B \in \mathcal{F}_5 \cup \mathcal{F}_6$ and $C \in \mathcal{F}_7 \cup \mathcal{F}_8$. Moreover, we can assume $B \in \mathcal{F}_5$. If $|A \cap B \cap W_1| = 4$ then for any 3-tuple or 5-tuple $C \cap W_1$ there is an element in W_1 belonging to exactly two of A, B and C. Thus, $A \cap W_1 = B \cap W_1$ and necessarily $A \cap W_1 = C \cap W_1$. It follows, $C \in \mathcal{F}_7$ and furthermore $B \cap W_2$ and $C \cap W_2$ are distinct triangles, since $B \cap W_2 \in \mathcal{H}_2$ and $C \cap W_2 \in \mathcal{H}_3$. Then their symmetric difference has a common element with $A \cap W_2$ which is a contradiction.

Case $[1,0]$. We may assume that A and B are in $\mathcal{F}_1$ and $C \in \mathcal{F}_5$. Then the triple $C \cap W_2$ does not cover $A \cap B \cap W_2$.

Case $[0,0]$. A, B and C belong to $\mathcal{F}_1$ and by Lemma 3.2 do not form a Δ-system.

This concludes the proof of the fact that $\mathcal{F}$ is 3-free, and so the proof of Lemma 4.2. $\square$

Proof of Theorem 1.4: Applying Lemma 3.1 with q sets instead of 2, the above construction gives for each n of the form $n = 14q$ a 3-free Sperner family showing $F(n,3) \geq (388^{1/14})^n > 1.53^n$. $\square$

4.3 The family $\mathcal{L}$ on 14 elements

We now define another Sperner 3-free family $\mathcal{L}$ of subsets of the 14-element set $W_1 \cup W_2 \cup \{x_1, x_2\}$. (Note: we will later take $\mathcal{L}$ to be on a ground set disjoint from that of $\mathcal{F}$.) As in Section 4.2, we shall use for $\mathcal{L}$ the same meaning for quadruples of the type $\langle i_1, i_2, j_1, j_2 \rangle$, where i_1 and i_2 will take values from $\{0,1,2,3\}$ and j_1, j_2 from $\{0,1\}$.

We put $\mathcal{L} = \bigcup_{t=1}^{8} \mathcal{L}_t$, which are defined by the following quadruples:

$$\begin{aligned}
\mathcal{L}_1 &= \langle 1,2,0,0 \rangle, \\
\mathcal{L}_2 &= \langle 2,1,0,0 \rangle, \\
\mathcal{L}_3 &= \langle 2,3,1,0 \rangle, \\
\mathcal{L}_4 &= \langle 3,2,0,1 \rangle, \\
\mathcal{L}_5 &= \langle 1,0,1,0 \rangle, \\
\mathcal{L}_6 &= \langle 0,1,0,1 \rangle, \\
\mathcal{L}_7 &= \langle 3,0,1,1 \rangle, \\
\mathcal{L}_8 &= \langle 0,3,1,1 \rangle.
\end{aligned}$$

Lemma 4.3 *The family $\mathcal{L}$ is Sperner, 3-free, and satisfies $|\mathcal{L}| = 352$.*

Proof of Lemma 4.3: We prove the lemma along the lines of the proof of Lemma 4.2.

One can check that $|\mathcal{L}_1| = |\mathcal{L}_2| = 60$, $|\mathcal{L}_3| = |\mathcal{L}_4| = 100$, $|\mathcal{L}_5| = |\mathcal{L}_6| = 6$, and $|\mathcal{L}_7| = |\mathcal{L}_8| = 10$, giving 352 in all.

To derive the Sperner property, observe first that each member of $\mathcal{L}_t$ has cardinality k_t, where $k_1 = k_2 = 8, k_3 = k_4 = 7, k_5 = k_6 = 6, k_7 = k_8 = 5$.

Notice that only the members of $\mathcal{L}_1$ and $\mathcal{L}_2$ do not meet $\{x_1, x_2\}$ and hence none of them contains any other member of $\mathcal{L}$. The members of $\mathcal{L}_3 \cup \ldots \cup \mathcal{L}_6$ have smaller intersection size with $\{x_1, x_2\}$ than those of $\mathcal{L}_7 \cup \mathcal{L}_8$. The members of $\mathcal{L}_3$ and $\mathcal{L}_4$ have smaller intersection size with W_1 than those of $\mathcal{L}_5$ and have smaller intersection size with W_2 than those of $\mathcal{L}_6$. Thus, $\mathcal{L}$ is Sperner.

Suppose that some members A, B and C of $\mathcal{L}$ form a Δ-system. As above, for $0 \leq p, q \leq 3$ we denote by case $[p, q]$ the case when x_1 belongs to exactly p many of A, B and C, and x_2 belongs to q of them. We also take into account the symmetry between p and q. In each case we shall find an element which belongs to exactly two of A, B and C, yielding a contradiction.

Case $[3, 3]$. Then A, B and C belong to $\mathcal{L}_7 \cup \mathcal{L}_8$. We may assume $A, B \in \mathcal{L}_7$. If C also belongs to $\mathcal{L}_7$, then the sets $A \cap W_1, B \cap W_1$ and $C \cap W_1$ form a Δ-system, a contradiction to Lemma 4.1. Let $C \in \mathcal{L}_8$. Then the elements of $W_1 \cap A \cap B$ do not belong to C.

Case $[3, 1]$. We may assume $A \in \mathcal{L}_7$. If both B and C belong to $\mathcal{L}_3$, then the set $W_2 \cap B \cap C$ is non-empty and disjoint from A. Let $B \in \mathcal{L}_5$. If C also belongs to $\mathcal{L}_5$, then $|W_1 \cap C \cap B| = 4$ and hence some element of this set is not in A. Finally, if $C \in \mathcal{L}_3$ then the symmetric difference between $B \cap W_1$ and $C \cap W_1$ has size at least two, and hence meets $A \cap W_1$.

Case $[3, 0]$. Assume first that A and B are in $\mathcal{L}_3$. Since the set $W_2 \cap B \cap A$ is non-empty, C also should be in $\mathcal{L}_3$. But by Lemma 3.1, $\mathcal{L}_3$ is Sperner and 3-free. Thus, we may assume that A and B are in $\mathcal{L}_5$. Then no other member of $\mathcal{L}_3 \cup \mathcal{L}_5$ covers $W_1 \cap A \cap B$.

Case $[1, 1]$. Assume first that A is in $\mathcal{L}_7 \cup \mathcal{L}_8$, for definiteness, in $\mathcal{L}_7$. Then both B and C are in $\mathcal{L}_1 \cup \mathcal{L}_2$, and hence the set $W_2 \cap B \cap C$ is non-empty and disjoint from A. Thus exactly one of A, B and C belongs to $\mathcal{L}_1 \cup \mathcal{L}_2$. We may assume that $A \in \mathcal{L}_1$, $B \in \mathcal{L}_3 \cup \mathcal{L}_5$, $C \in \mathcal{L}_4 \cup \mathcal{L}_6$. Note that in any case, the symmetric difference between $B \cap W_1$ and $C \cap W_1$ has size at least two, and hence meets $A \cap W_1$.

Case $[1, 0]$. We may assume that both B and C are in $\mathcal{L}_1 \cup \mathcal{L}_2$. If $A \in \mathcal{L}_5$ then the set $W_2 \cap B \cap C$ is non-empty and disjoint from A. Let $A \in \mathcal{L}_3$. If, say, $B \in \mathcal{L}_2$, then the symmetric difference between $A \cap W_2$ and $C \cap W_2$ has size at least two, and hence meets $B \cap W_2$. If, finally, both B and C are in $\mathcal{L}_1$, then the set $B \cap C \cap W_1$ has size at least four, and hence is not covered by $A \cap W_1$.

Case $[0, 0]$. We may assume that A and B are in $\mathcal{L}_1$. If $C \in \mathcal{L}_2$, then the triple $C \cap W_1$ does not cover $A \cap B \cap W_1$, and so $C \in \mathcal{L}_1$. But by Lemma 3.1, $\mathcal{L}_1$ is Sperner and 3-free. $\Box$

4.4 Another product lemma

The following lemma is a relative of Theorem 2 in [1].

Lemma 4.4 . *Let $\mathcal{A}$ and $\mathcal{B}$ be Sperner 3-free families on disjoint ground sets A and B, respectively. For $a \in A$ and $b \in B$, set $\mathcal{A}_a = \{C \in \mathcal{A} : a \in C\}$, $\mathcal{B}_b = \{D \in \mathcal{B} : b \in D\}$, $\overline{\mathcal{A}}_a = \mathcal{A} \setminus \mathcal{A}_a$, and $\overline{\mathcal{B}}_b = \mathcal{B} \setminus \mathcal{B}_b$. Let $\mathcal{G}_1 = \{(C \setminus \{a\}) \cup D : C \in \mathcal{A}_a, D \in \overline{\mathcal{B}}_b\} and \mathcal{G}_2 = \{C \cup (D \setminus \{b\}) : C \in \overline{\mathcal{A}}_a, D \in \mathcal{B}_b\}$. Then for $\mathcal{G} = \mathcal{G}(\mathcal{A}, a, \mathcal{B}, b) = \mathcal{G}_1 \cup \mathcal{G}_2$, the following hold:*

(i) $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$;

(ii) $\mathcal{G}$ is Sperner;

(iii) $\mathcal{G}$ is a 3-free family on the ground set $(A \cup B) \setminus \{a, b\}$.

Proof: Let $M_i \in \mathcal{G}_i$, $i = 1, 2$, $M_i \cap A = C_i$, $M_i \cap B = D_i$. Assume that $M_1 \supset M_2$. Then $C_1 \supset C_2$, which is impossible because, by definition, $C_1 \cup \{a\}$ and C_2 are members of the Sperner family $\mathcal{G}_1$, implying (i). Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are Sperner, this implies (ii).

Now assume that some distinct members M_1, M_2 and M_3 of $\mathcal{G}$ (where $M_i \cap A = C_i$, $M_i \cap B = D_i$) form a Δ-system. Due to the symmetry between $\mathcal{G}_1$ and $\mathcal{G}_2$, it is enough to consider the following cases.

CASE 1. All M_1, M_2 and M_3 are members of $\mathcal{G}_1$. Then D_1, D_2 and D_3 should form a Δ-system, too (maybe with repetition of members). Since $\mathcal{G}_2$ is Sperner and 3-free, $D_1 = D_2 = D_3$ is necessary. Analogously, C_1, C_2 and C_3 (and hence also $C_1 \cup \{a\}, C_2 \cup \{a\}$ and $C_3 \cup \{a\}$) form a Δ-system, as well. Again, we get $C_1 = C_2 = C_3$. Thus, $M_1 = M_2 = M_3$, a contradiction.

CASE 2. $M_1, M_2 \in \mathcal{G}_1$, $M_3 \in \mathcal{G}_2$. As in Case 1, D_1, D_2 and D_3 should form a Δ-system, too (maybe with repetition of members). Then D_1, D_2 and $D_3 \cup \{b\}$ are members of $\mathcal{G}_2$ and form a Δ-system, as well, but b belongs only to $D_3 \cup \{b\}$. This is impossible for the Sperner and 3-free $\mathcal{G}_2$. $\Box$

4.5 The families $\mathcal{Q}$ and $\mathcal{R}$

We first construct from $\mathcal{F}$ and $\mathcal{L}$ a new family $\mathcal{Q}$ on 26 vertices. Let $a \in W_1$ and $b \in W_2$ be some elements of our 14-element set X. It is routine to verify that, in terms of Lemma 4.4,

$$|\mathcal{F}_b \cap \mathcal{F}_a| = 150,\ |\mathcal{F}_b \cap \overline{\mathcal{F}}_a| = |\overline{\mathcal{F}}_b \cap \mathcal{F}_a| = 95,\ |\overline{\mathcal{F}}_b \cap \overline{\mathcal{F}}_a| = 48, \quad (8)$$

and

$$|\mathcal{L}_{x_1} \cap \mathcal{L}_{x_2}| = 20, \ |\mathcal{L}_{x_1} \cap \overline{\mathcal{L}}_{x_2}| = 106, \ |\overline{\mathcal{L}}_{x_1} \cap \mathcal{L}_{x_2}| = 106, \ |\overline{\mathcal{L}}_{x_1} \cap \overline{\mathcal{L}}_{x_2}| = 120. \quad (9)$$

Let $\mathcal{F}$ and $\mathcal{L}$ have disjoint 14-element ground sets $X(1)$ and $X(2)$, respectively, where now for each $i = 1, 2$, $W_1(i)$, $W_2(i)$, $x_1(i)$, $x_2(i)$, $a(i)$, and $b(i)$ denote the corresponding copies of W_1, W_2, x_1, x_2, a, and b in $X(i)$. We define

$$\mathcal{Q} = \mathcal{G}(\mathcal{F}, b(1), \mathcal{L}, x_1(2)).$$

By Lemma 4.4, $\mathcal{Q}$ is Sperner and 3-free. In anticipation of defining another family $\mathcal{R}$, we make some preliminary calculations. By (8) and (9),

$$\begin{aligned}
|\mathcal{Q}| &= 245 \cdot 226 + 143 \cdot 126 = 73388; \\
|\mathcal{Q}_{a(1)}| &= 150 \cdot 226 + 95 \cdot 126 = 45870; \\
|\overline{\mathcal{Q}}_{a(1)}| &= 73388 - 45870 = 27518; \\
|\overline{\mathcal{Q}}_{x_2(2)}| &= 245 \cdot 120 + 143 \cdot 106 = 44558; \\
|\mathcal{Q}_{x_2(2)}| &= 73388 - 44558 = 28830.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\mathcal{Q}_{a(1)} \cup \mathcal{Q}_{x_2(2)}| &= 150 \cdot 106 + 95 \cdot 20 = 17800; \\
|\overline{\mathcal{Q}}_{a(1)} \cup \mathcal{Q}_{x_2(2)}| &= 95 \cdot 106 + 48 \cdot 20 = 11030; \\
|\mathcal{Q}_{a(1)} \cup \overline{\mathcal{Q}}_{x_2(2)}| &= 150 \cdot 120 + 95 \cdot 106 = 28070; \\
|\overline{\mathcal{Q}}_{a(1)} \cup \overline{\mathcal{Q}}_{x_2(2)}| &= 95 \cdot 120 + 48 \cdot 106 = 16488.
\end{aligned}$$

We now define the family $\mathcal{R}$ on a ground set of 50 vertices. Let $\mathcal{Q}(1)$ and $\mathcal{Q}(2)$ be two copies of $\mathcal{Q}$ on disjoint ground sets. We define

$$\mathcal{R} = \mathcal{G}(\mathcal{Q}(1), a(1), \mathcal{Q}(2), x_2(2)).$$

By Lemma 4.4, $\mathcal{R}$ is Sperner and 3-free.

Let w be the copy of $a(1)$ on the ground set of $\mathcal{Q}(1)$ and x be the copy of $x_2(2)$ on the ground set of $\mathcal{Q}(2)$. By the above calculations,

$$\begin{aligned}
|\mathcal{R}_w| &= 28070 \cdot 45870 + 17800 \cdot 27518 = 1\,777\,391\,300, \\
|\overline{\mathcal{R}}_w| &= 16488 \cdot 45870 + 11030 \cdot 27518 = 1\,059\,828\,100, \\
|\mathcal{R}_x| &= 17800 \cdot 44558 + 11030 \cdot 28830 = 1\,111\,127\,300, \\
|\overline{\mathcal{R}}_x| &= 28070 \cdot 44558 + 16488 \cdot 28830 = 1\,726\,092\,100,
\end{aligned}$$

and so

$$|\mathcal{R}| = 1\,111\,127\,300 + 1\,726\,092\,100 = 2\,837\,219\,400.$$

We remark that, as in the proof of Theorem 1.4, the construction of $\mathcal{R}$ gives for each n of the form $n = 50q$ a 3-free Sperner family showing $F(n,3) \geq (2\,837\,219\,400^{n/50}) > 1.545^n$, however, we can do somewhat better.

4.6 The proof of Theorem 1.5

Proof of Theorem 1.5: For $j = 1, 2, \ldots$, we construct a Sperner and 3-free family $\mathcal{R}^j$ of cardinality at least 1.551^{48j} with the ground set D^j, $|D^j| = 48j + 2$. We put $\mathcal{R}^1 = \mathcal{R}$ and by above calculations, observe that $|\mathcal{R}^1| = 2\,837\,219\,400 > 1.551^{48}$.

Suppose that $\mathcal{R}^{j-1}$ has been constructed on the ground set D^{j-1}. Let z be any element of D^{j-1}, and fix a copy of $\mathcal{R}$ on a ground set disjoint from D^{j-1}. Using Lemma 4.4, we will take a certain product of $\mathcal{R}^{j-1}$ with the new copy of $\mathcal{R}$, depending on certain vertices.

CASE 1. If $|\mathcal{R}_z^{j-1}| \geq 0.5|\mathcal{R}^{j-1}|$ then put $\mathcal{R}^j = \mathcal{G}(\mathcal{R}^{j-1}, z, \mathcal{R}, x)$. By construction and the induction assumption, $|D^j| = |D^{j-1}| + 48 = 48j + 2$ and

$$\begin{aligned} |\mathcal{R}^j| &= 1\,726\,092\,100 \cdot |\mathcal{R}_z^{j-1}| + 1\,111\,127\,300 \cdot |\overline{\mathcal{R}}_z^{j-1}| \\ &\geq |\mathcal{R}^{j-1}|(0.5 \cdot 1\,726\,092\,100 + 0.5 \cdot 1\,111\,127\,300) \\ &\geq 1.551^{48(j-1)} \cdot 0.5 \cdot 2\,837\,219\,400 \ > \ 1.551^{48j}. \end{aligned}$$

CASE 2. If $|\mathcal{R}_z^{j-1}| < 0.5|\mathcal{R}^{j-1}|$ then put $\mathcal{R}^j = \mathcal{G}(\mathcal{R}^{j-1}, z, \mathcal{R}, w)$. Similar to Case 1, $|D^j| = 48j + 2$ and

$$\begin{aligned} |\mathcal{R}^j| &= 1\,059\,828\,100 \cdot |\mathcal{R}_z^{j-1}| + 1\,777\,391\,300 \cdot |\overline{\mathcal{R}}_z^{j-1}| \\ &\geq |\mathcal{R}^{j-1}|(0.5 \cdot 1\,059\,828\,100 + 0.5 \cdot 1\,777\,391\,300) \\ &\geq 1.551^{48(j-1)} \cdot 0.5 \cdot 2\,837\,219\,400 \ > \ 1.551^{48j}. \ \square \end{aligned}$$

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