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Near Sets.

General Theory About Nearness of Objects

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Abstract

The problem considered in this paper is the approximation of sets of perceptual objects that are qualitatively near each other. A perceptual object is either something presented to the senses or knowable by the mind. Objects that have the same appearance are considered *qualitatively near* each other, *i.e.*, objects with matching descriptions. The term *approximate* means *very near, in position or in character*. The solution to the problem of approximating sets of perceptual objects results from a generalization of the approach to the classification of objects introduced by Zdzisław Pawlak's during the early 1980s. This generalization leads to the introduction of near sets. In addition, a formal explanation of the predicate *near* relative to near objects, near sets and nearness approximation spaces is given. The contribution of this paper is a formal basis for the discovery of perceptual objects that are qualitatively near each other.

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1 Introduction

The problem considered in this article is the approximation of sets of perceptual objects that are qualitatively near each other. Perceptual objects that have the same appearance are considered *qualitatively near* each other, *i.e.*, objects with matching descriptions. A *description* is a tuple of values of functions representing features of an object. For simplicity, assume the description of an object consists of one function value. For example, let $b \in L,\ b' \in L'$ be books contained in two libraries $L,\ L'$ and $\phi(b)$ = number of pages in b, where book length is a feature of a book and ϕ is a sample function representing book length. Then book b is near book b' if $\phi(b) = \phi(b')$.

Sets $X,\ X'$ are considered near each other if, and only if there exist objects $x \in X,\ x' \in X'$ such that $x,\ x'$ have matching descriptions. In that case, $X,\ X'$ are called near sets. This means that the predicate *is near* applied to either objects or sets is defined relative to description and not in terms of set membership. In effect, it is possible to have $X \cap X' = \emptyset$ (non-intersecting sets) and, yet, the assertion $x \in X$ is near $x' \in X'$ is true if, only if $x,\ x'$ having matching descriptions to some degree. Notice that if we replace X' by X, we arrive at a special case where a single set is considered a near set. That is, a single set X containing two or more objects that have matching descriptions is considered a near set.

The phrase *qualitatively near* is close to the usual understanding of the adjective *similar* [15]. Insight into the perception of near objects comes from Zdzisław Pawlak's work on classification of objects [20, 21, 22] and from Ewa Orłowska's observation about approximation spaces as formal counterparts of perception [16]. The formal theory of near sets presented in this paper is related to the study of information theory considered in the context of information systems in rough set theory [5] and the recent study of dialogue considered in the context of approximation spaces [6]. The focus of this paper is on the possibility of perceptual synthesis[1], an interpretation of perception suggested by Rabindranath Tagore [47]. In the context of near sets, this synthesis results from an extension of the approach to approximation spaces introduced by Zdzisław Pawlak and Ewa Orłowska during the 1980s.

An understanding of perception either by humans or imitated by thinking machines entails a consideration of the appearances of objects characterized by functions representing object features. In this paper, a formal explanation of the predicate *near* is given relative to near objects, near sets and nearness approximation spaces. Recall that in grammar and logic, a *predicate* is something that is asserted about the subject of a sentence or concerning the argument of a proposition. Hence, in some sense, we can assert that one perceptual object is *near* another object or one set of perceptual objects is *near* another set of perceptual objects.

In this paper, the term *near* is used to characterize either the correspondence between perceptual objects or the affinity between sets containing perceptual objects with matching descriptions. A *perceptual object* is either something presented to the senses or knowable by the mind [15]. This can be explained informally in the following way.

Perceptual objects are considered *near* each other if the objects have similar descriptions to some degree. For example, two peas in a pod are considered

[1] *i.e.*, perception on the level of classes rather than on the level of individual objects.

near each other if they have approximately the same colour or shape or weight independent of the relative position of the peas. A set of perceptual objects is a near set if it contains objects that are near each other. Hence, any equivalence class containing perceptual objects with matching descriptions is a near set. Again, for example, any set of buildings in the same village is a near set if one considers location as part of the description of the buildings. Any non-empty rough set contains one or more equivalence classes, *i.e.* sets containing objects with matching descriptions. Hence, any rough set is a near set.

By contrast with work on proximity spaces by Efremovič [9], this paper points to a more general understanding of nearness that is not restricted to spatial proximity. A realization of the proposed approach to the perception of objects can be found in approximate adaptive learning introduced in [31] and a number of studies of biologically-inspired machine learning [14, 25, 27, 29, 31, 32, 35, 33, 34, 37, 38], nearness in approximation spaces [28, 29], and the near set approach to set approximation [11, 23, 24]. The contribution of this article is a presentation of a formal basis for the discovery of perceptual objects that are near each other.

This paper is organized as follows. A brief overview of the approach to object description is given in Sect. 2. The notation, definitions and theorems concerning the nearness of objects and set approximation are covered in Sect. 3 and Sect. 4. Nearness approximation spaces are presented in Sect. 5.

2 Object Description

Table 1: Description Symbols

Symbol	Interpretation
$\Re$	Set of real numbers,
$\mathcal{O}$	Set of perceptual objects,
X	$X \subseteq \mathcal{O}$, set of sample objects,
x	$x \in \mathcal{O}$, sample object,
$\mathcal{F}$	A set of functions representing object features,
B	$B \subseteq \mathcal{F}$,
ϕ	$\phi : \mathcal{O} \rightarrow \Re^L$, object description,
L	L is a description length,
i	$i \leq L$,
ϕ_i	$\phi_i \in B$, where $\phi_i : \mathcal{O} \longrightarrow \Re$, probe function,
$\phi(x)$	$\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \ldots, \phi_i(x), \ldots, \phi_L(x))$.

Objects are known by their descriptions. An *object description* is defined by means of a tuple of function values $\phi(x)$ associated with an object $x \in X$ (see (1)). The important thing to notice is the choice of functions $\phi_i \in B$ used to describe an object of interest. Assume that $B \subseteq \mathcal{F}$ (see Table 1) is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\phi_i \in B$, where $\phi_i : \mathcal{O} \longrightarrow \Re$. In combination, the functions representing object features provide a basis for an *object description* $\phi : \mathcal{O} \longrightarrow \Re^L$, a vector containing measurements (returned values) associated with each functional value $\phi_i(x)$ in (1), where the description length $|\phi| = L$.

$$\textbf{Object Description}: \quad \phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_i(x), \ldots, \phi_L(x)). \quad (1)$$

The intuition underlying a description $\phi(x)$ is a recording of measurements from sensors, where each sensor is modelled by a function ϕ_i.

2.1 Sample Behaviour Description

Table 2: Sample ethogram

x_i	s	a	$p(s,a)$	r	d
x_0	0	1	0.1	0.75	1
x_1	0	2	0.1	0.75	0
x_2	1	2	0.05	0.1	0
x_3	1	3	0.056	0.1	1
x_4	0	1	0.03	0.75	1
x_5	0	2	0.02	0.75	0
x_6	1	2	0.01	0.9	1
x_7	1	3	0.025	0.9	0

By way of illustration, consider the description of the behaviour observable in biological organisms. For example, a behaviour can be represented by a tuple

$$(s, a, p(s,a), r)$$

where $s, a, p(s,a), r$ denote organism functions representing state, action, action preference in a state, and reward for an action, respectively. A reward r is observed in state s and results from an action a performed in the previous state. The preferred action a in state s is calculated using

$$p(s,a) \leftarrow p(s,a) + \beta\delta(r,s),$$

where β is the actor's learning rate and $\delta(r,s)$ is used to evaluate the quality of action a (see [27]). In combination, tuples of behaviour function values form the following description of an object x relative to its observed behaviour:

$$\textbf{Organism Behaviour}: \quad \phi(x) = (s(x), a(x), r(x), V(s(x))).$$

Table 2 is an example of what is known as a rough ethogram [36, 39], a tabulation of observed behaviours of an organism.

3 Nearness of Objects

Approximate, *a* [L. *approximat-us* to draw near to.]
A. *adj.*
1. Very near, in position or in character;
closely situated; nearly resembling.
–Oxford English Dictionary, 1933.

Table 3: Relation and Partition Symbols

Symbol	Interpretation
$\sim_B$	$\{(x, x') \mid f(x) = f(x')\ \forall f \in B\}$, indiscernibility relation,
$[x]_B$	$[x]_B = \{x' \in X \mid x' \sim_B x\}$, elementary set (class),
$\mathcal{O}/\sim_B$	$\mathcal{O}/\sim_B = \{[x]_B \mid x \in \mathcal{O}\}$, quotient set,
ξ_B	Partition $\xi_B = \mathcal{O}/\sim_B$,
$\Delta\phi_i$	$\Delta\phi_i = \phi_i(x') - \phi_i(x)$, probe function difference.

Sample objects $X \subseteq \mathcal{O}$ are near each other if, and only if the objects have similar descriptions. Recall that each ϕ[2] defines a description of an object (see Table 1). Then let $\Delta\phi_i$ denote

$$\Delta\phi_i = \phi_i(x') - \phi_i(x),$$

where $x, x' \in \mathcal{O}$. The difference $\Delta\phi$ leads to a definition of the indiscernibility relation $\sim_B$ introduced by Zdzisław Pawlak [21] (see Def 3.1).

Definition 3.1 Indiscernibility Relation
Let $x, x' \in \mathcal{O}, B \in \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \forall \phi_i \in B \,\centerdot\, \Delta\phi_i = 0\},$$

where $i \leq |\phi|$ (description length).

Definition 3.2 Nearness Description Principle (NDP)
Let $B \subseteq \mathcal{F}$ be a set of functions representing features of objects $x, x' \in \mathcal{O}$. Objects x, x' are minimally near each other if, and only if there exists $\phi_i \in B$ such that $x \sim_{\{\phi_i\}} x'$, i.e., $\Delta\phi_i = 0$.

[2]In a more general setting that includes data mining, ϕ_i would be defined to allow for non-numerical values, *i.e.*, let $\phi_i : X \longrightarrow V$, where V is the *value set* for the range of ϕ_i [40]. This more general definition of $\phi_i \in \mathcal{F}$ is also better in setting forth the algebra and logic of near sets after the manner of the algebra and logic of rough sets [1, 2, 4, 7, 40]. Real-valued probe functions are used in object descriptions in this article because we have science and engineering applications of near sets in mind.

In effect, objects x, x' are considered *minimally near* each other whenever there is at least one probe function $\phi_i \in B$ so that $\phi_i(x) = \phi_i(x')$. A *probe function* can be thought of as a model for a sensor (see, *e.g.*, [8, 10, 19, 26]). Then ϕ_i constitutes a minimum description of the objects x, x' that makes it possible for us to assert that x, x' are near each other. Ultimately, there is interest in identifying the probe functions that lead to partitions with the highest information content. The nearness description principle (NDP) differs markedly from minimum description length (MDL) proposed by Jorma Rissanen [41]. MDL deals with a set $X = \{x_i \mid \ i = 1, \dots\}$ of possible data models and a set Θ of possible probability models. By contrast, NDP deals with a set X that is the domain of a description $\phi : X \longrightarrow \Re^L$ and the discovery of at least one probe function $\phi_i(x)$ in a particular description $\phi(x)$ used to identify similar objects in X. The term *similar* is used here to denote the presence of objects $x, x' \in X$ and at least one ϕ_i in object description ϕ, where $x \sim_{\phi_i} x'$. In that case, objects x, x' are said to be similar. This leads to a feature selection method, where one considers combinations of n probe functions r in searching for those combinations of probe functions that lead to partitions with the highest information content (see, *e.g.*, [30]).

Observation 3.3 Near Objects in a Class
Let $\xi_B = \mathcal{O}/\sim_B$ denote a partition of $\mathcal{O}$. Let $[x]_B \in \xi_B$ denote an equivalence class. Assume $x, x' \in [x]_B$. From Table 3 and Def. 3.1, we know that for each $\phi_i \in B, \Delta\phi_i = 0$. Hence, from Def. 3.2, x, x' are near objects.

Theorem 3.4 *The objects in a class $[x]_B \in \ \xi_B$ are near objects.*

Proof 3.4.1 *The nearness of objects in a class follows from Obs. 3.3.* □

Definition 3.5 Measure of Object Nearness
Let $B \subseteq \mathcal{F}$ be a set of functions representing features of objects in $\mathcal{O}$. Let $X,\ X' \subseteq \mathcal{O}$ denote a set of objects of interest and set of test objects, respectively. Let $\phi_i \in B$, where $i \leq |B|$. Let $\mu_X^B : \wp(\mathcal{O}) \longrightarrow [0,1]$ denote a capacity function defined by

$$\mu_X^B(X') = \frac{|\{\phi_i \in B \mid \ x \in X, x' \in X' \centerdot x \sim_{\phi_i} x'\}|}{|B|}.$$

Example 3.6 Sample Near Objects *Let $\mathcal{O}$ denote a set of sample pieces of furniture. Let $X_{Thai}, X_{Wpg} \subseteq \mathcal{O}$ denote a set of tables in Bankok, Thailand (Thai) and a set of chairs in Winnipeg (Wpg), Manitoba, respectively. Assume that probe function $\phi_i \in B \subseteq \mathcal{F}$ represents a feature of furniture in $\mathcal{O}$. Further, assume $i \leq |B| = 5$ and that table $x \in X_{Thai}$ and chair $x' \in X_{Wpg}$ have a matching description only in terms of ϕ_i. That is, in the case where $x \sim_{\phi_i} x'$ for a Thai table x and a Wpg chair x', we have*

$$\mu_{X_{Thai}}^B(X_{Wpg}) = \frac{1}{5}.$$

3.1 Near Sets

The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects X, X' are considered near each other if the sets contain objects with at least partial matching descriptions.

Definition 3.7 Near Sets
Let $X,\ X' \subseteq \mathcal{O},\ B \subseteq \mathcal{F}$. Set X is near X' if, and only if there exists $x \in X, x' \in X', \phi_i \in B$ such that $x \sim_{\{\phi_i\}} x'$.

Object recognition problems, especially in images [3, 11], and the problem of the nearness of objects have motivated the introduction of near sets (see, *e.g.*, [23, 28]).

Remark 3.8 *If X is near X', then X is a near set relative to X' and X' is a near set relative to X. Notice that if we replace X' by X in Def. 3.7, this leads to what is known as* reflexive nearness*.*

Definition 3.9 Reflexive Nearness
If $x, x' \in X$ and x is near x', then by Def. 3.7 X is a near set relative to itself. In fact, X is a near set.

Observation 3.10 Class as a Near Set
By definition, a class $[x]_B$ in a partition ξ_B is a set of objects having matching descriptions (set Table 3), i.e., if $x, x' \in [x]_B$, then $x \sim_B x'$.

Theorem 3.11 *A class in a partition ξ_B is a near set.*

Proof 3.11.1 *From Obs. 3.10 and from Def. 3.9, we know that a class $[x]_B \in \xi_B$ is a near set.* □

Affinities between objects of interest in the set $X \subseteq \mathcal{O}$ can be discovered by considering the relation between X and objects in elementary sets in partition $\mathcal{O}/\sim_B$. Approximation of the set X begins by determining which elementary sets $[x]_B \subseteq \mathcal{O}/\sim_B$ are subsets of X.

Observation 3.12 Near Partition of Objects
Assume that $\sim_B$ (Def. 3.1) is an equivalence relation that defines a partition $\xi_B = \mathcal{O}/\sim_B$, i.e., *a partition of $\mathcal{O}$. Let $[x]_B \in \xi_B$ denote an equivalence class. This means that $\forall \phi_i \in B,\ x \sim_B x'\ \forall x,\ x' \in [x]_B$.*

Theorem 3.13 *A partition ξ_B is a near set.*

Proof 3.13.1 *From Obs. 3.12 and Theorem 3.11, the classes in partition ξ_B are near sets. In effect, ξ_B contains objects that are near each other. Hence, ξ_B is a near set.* □

Definition 3.14 Hierarchy of Near Sets
If X', $X'' \subseteq X$ and X', X'' are near sets, then by extension of Def. 3.9 X is a near set.

Observation 3.15 Inherited Nearness
By implication from Def 3.14, any set that contains a near set is itself considered a near set.

Theorem 3.16 *A set containing a near set is itself a near set.*

Proof 3.16.1 *Assume a set X contains a near set. From Def. 3.14 and Obs. 3.15, X is a near set.* □

3.2 Near Sets and Verisimilitude

> The better theory is the more precise description of the [object] it provides.
> –Ewa Orłowska, Studia Logica, 1990.

Truth values can be associated with assertions about the nearness of individual objects as well as sets of objects. Let $(\mathcal{O}, \mathcal{F})$ denote an information system, where $B \subseteq \mathcal{F}$. Consider, for example, the following strong nearness condition: sets $X, X' \subseteq \mathcal{O}$ are strongly near each other in the case where there exists objects $x \in X, x' \in X$ such that $x \sim_B x' \; \forall \phi_i \in B$. One might postulate the concept of *strong nearness* and evaluate the verisimilitude[3] of an assertion about the nearness of sets X, X' in terms of whether the sets satisfy the strong nearness condition. This is in keeping with the notion of verisimilitude useful in the comparison of theories resulting from learning concepts [18].

4 Fundamental Approximation Space

This section presents a number of near sets resulting from the approximation of one set by another set. Approximations are carried out within the context of a fundamental approximation space $FAS = (\mathcal{O}, \mathcal{F}, \sim_B)$, where $\mathcal{O}$ is a set of perceived objects, $\mathcal{F}$ is a set of probe functions representing object features, and $\sim_B$ is an indiscernibility relation defined relative to $B \subseteq \mathcal{F}$. The space FAS is considered fundamental because it provided a framework for the original rough set theory [21]. It has also been observed that an approximation space is the formal counterpart of perception [16]. Approximation starts with the partition ξ_B of $\mathcal{O}$ defined by the relation $\sim_B$. Next, any set $X \subseteq \mathcal{O}$ is approximated by considering the relation between X and the classes $[x]_B \in \xi_B, \; x \in \mathcal{O}$. To see this, consider first the lower approximation of a set.

[3] *i.e.*, the appearance of being true [15].

4.1 Lower Approximation of a Set

From the first dawn of life, all organic beings are found
to resemble each other in descending degrees,
so that they can be classed in groups under groups.
–Charles Darwin, On the Origin of the Species, XIII, 1859.

Table 4: Approximation Notation

Symbol	Interpretation
$(\mathcal{O}, \mathcal{F}, \sim_B)$	Fundamental approximation space (FAS), $B \subseteq \mathcal{F}$,
B_*X	$\bigcup_{x:[x]_B \subseteq X} [x]_B$, B-lower approximation of X,
B^*X	$\bigcup_{x:[x]_B \cap X \neq \emptyset} [x]_B$, B-upper approximation of X,
$Bnd_B X$	$Bnd_B X = B^*X \setminus B_*X = \{x \mid x \in B^*X \text{ and } x \notin B_*X\}$.

This section and its sequel introduce the basic approach to approximation introduced by Zdzisław Pawlak (see, *e.g.*, [21, 22]) and elaborated by others during the early 1980s (see, *e.g.*, [13, 16, 17]). The indiscernibility relation $\sim_B$ defines the partition of a set of objects $\mathcal{O}$ into elementary sets (classes) containing objects with matching descriptions. The net result of this partition is the classification of the objects in $\mathcal{O}$. Each elementary set $[x]_B \in \mathcal{O}/\sim_B$ represents an information granule, a distillation of our knowledge represented by the matching descriptions for all objects in $[x]_B$ relative to the functions in B.

Affinities between objects of interest in the set $X \subseteq \mathcal{O}$ and classes in the partition ξ_B can be discovered by identifying those classes that have objects in common with X. Approximation of the set X begins by determining which elementary sets $[x]_B \in \mathcal{O}/\sim_B$ are subsets of X. This discovery process leads to the construction of what is known as the B-lower approximation of $X \subseteq \mathcal{O}$, which is denoted by B_*X in (2).

$$B_*X = \bigcup_{x:[x]_B \subseteq X} [x]_B. \tag{2}$$

In effect, if B_*X (B-lower approximation of X) is not empty, then the objects in each class in B_*X have descriptions (*i.e.*, function values for each function $f \in B$) matching the descriptions of the corresponding objects in X.

Observation 4.1 Lower Approximation as a Near Set
The lower approximation B_*X of a set X is populated with classes that are

subsets of X. From Theorem 3.11, each class is a near set. From Def. 3.14, we also know that X can be viewed as a hierarchy of near sets represented by one or more classes $[x]_B \subseteq B_*X \subseteq X$.

Lemma 4.2 *The lower approximation B_*X of a set X is a near set.*

Theorem 4.3 *If a set X has a non-empty lower approximation B_*X, then X is a near set.*

Proof 4.3.1 *Given a set X with a non-empty lower approximation B_*X, then each class in B_*X is a subset of X. From Theorem 3.11, each class is a near set. It follows from Obs 4.1, Lemma 4.2 and Theorem 3.16 that X is a near set.* □

4.2 Upper Approximation of a Set

To begin, assume that $X \subset \mathcal{O}$, where X contains perceived objects that are in some sense interesting. Also assume that B contains functions representing features of objects in $\mathcal{O}$. A B-upper approximation of X is defined in (3).

$$B^*X = \bigcup_{x:[x]_B \cap X \neq \emptyset} [x]_B. \tag{3}$$

If we start with a set $\mathcal{O}$ containing perceived objects, then B^*X is interpreted relative to a set $X \subseteq \mathcal{O}$. By contrast with a B-lower approximation, the B-upper approximation B^*X is a collection of equivalence classes $[x]_B \in \mathcal{O}/\sim_B$, where each class in $B * X$ contains at least one object with a description that matches the description of an object in X. Notice that B_*X is always a subset of B^*X. Notice, also, that there may or may not be one or more equivalence classes $[x]_B \in \mathcal{O}/\sim_B$ that are not subsets of B^*X.

Observation 4.4 Upper Approximations Are Near Sets
From Def. 3.14 and Theorem 4.5, an upper approximation B^*X of a set X is a near set, since B^*X contains one or more classes that are near sets. By definition, B^*X and X have one or more objects in common and the common objects have matching descriptions.

Theorem 4.5 *The upper approximation B^*X and the set X are near sets.*

Proof 4.5.1 *Given the upper approximation B^*X and the set X, we know from Theorem 3.11 that the classes in B^*X are near sets. Hence, from Theorem 3.16, B^*X is a near set. From Obs. 4.4, we know that B^*X and the set X have common objects with matching descriptions. Hence, from Def. 3.7, we can conclude that B^*X and the set X are near sets.* □

The question concerning the extent that a class $[x]_B \in B^*X$ belongs to X is answered in an approximate ("nearly") way, depending on the proportion of the objects in $[x]_B$ that are included in X[4]. The next problem to consider is the extent that B^*X approximates (is near to) the set X. There are a number of solutions to this problem.

4.3 Boundary Region

The lower and upper approximations of a set lead naturally to the notion of a boundary region of an approximation defined in terms of set difference. Let $Y,\ Y' \subseteq \mathcal{O}$. The notation $Y \setminus Y'$ denotes set difference. Put

$$Y \setminus Y' = \{x \ \mid \ x \in Y \text{ and } x \notin Y'\}.$$

Let $Bnd_B X$ denote the boundary region of an approximation defined as shown in (4).

$$Bnd_B X = B^*X \setminus B_*X = \{x \ \mid \ x \in B^*X \text{ and } x \notin B_*X\}. \tag{4}$$

Observation 4.6 Rough Set with a Non-Empty Boundary
A set X is considered rough *whenever the boundary of an approximation is not empty,* i.e., *X is roughly or approximately known relative to the functions in B if, and only if $Bnd_B X \neq \emptyset$,* i.e., $|Bnd_B X| > 0$*. In that case, X is a rough set. Assume $Bnd_B X \neq \emptyset$. Then the lower approximation B_*X is a proper subset of upper approximation B^*X. From Lemma 4.2, we also know that the lower approximation B_*X of the set X is a near set. Each class in B_*X is a subset of X. Hence, X is a near set (see Theorem 4.8, case 1).*

Observation 4.7 Near Set with an Empty Boundary
It should also be observed that whenever $Bnd_B X = \emptyset$, this means that $|Bnd_B X| = 0$, $B_*X = B^*X$ and $B_*X \subseteq X$. From this, we know that B_*X and X share objects that have matching descriptions. Hence, X is a near set (see Theorem 4.8, case 2).

Theorem 4.8 Fundamental Near Set Theorem
A set X with an approximation boundary $|Bnd_B X| \geq 0$ is a near set.

Proof 4.8.1 *There are two cases to consider.*

1. *$|Bnd_B X| > 0$. Given a set $X \subset \mathcal{O}$ that has been approximated with a non-empty boundary, this means that X has a lower approximation $B_*X \subsetneq B^*X$* (i.e., *B_*X is a proper subset of B^*X). The classes $[x]_B \in B_*X$ are members of the partition $\xi_{\mathcal{O}}$. From (2), the set X contains the classes in B_*X. From Theorem 4.3, X is a near set.*

[4]This is essentially the approach that was originally suggested in [13, 21].

2. $|Bnd_B X| = 0$. *Given a set X that has been approximated with an empty boundary, this means that* $B_*X = B^*X$ *and from (2)* $B_*X \subseteq X$. *It follows from Theorem 4.3 and Obs. 4.7 that X is a near set.*

□

5 Nearness Approximation Spaces

Table 5: Nearness Approximation Space Symbols

Symbol	Interpretation
B	$B \subseteq \mathcal{F}$,
B_r	$r \leq \|B\|$ probe functions in B,
$\sim_{B_r}$	Indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid \ x \sim_{B_r} x'\}$, equivalence class,
$\mathcal{O}/\sim_{B_r}$	$\mathcal{O}/\sim_{B_r} = \left\{[x]_{B_r} \mid \ x \in \mathcal{O}\right\}$, quotient set,
$\xi_{\mathcal{O},B_r}$	Partition $\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r}$,
r	$\binom{\|B\|}{r}$, *i.e.*, $\|B\|$ functions $\phi_i \in B$ taken r at a time,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O},B_r} \mid \ B_r \subseteq B\}$, set of partitions,
ν_{N_r}	$\nu_{N_r} : \mathcal{P}(\mathcal{O}) \times \mathcal{P}(\mathcal{O}) \longrightarrow [0,1]$, overlap function,
$N_r(B)_*X$	$N_r(B)_*X = \bigcup_{x:[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^*X$	$N_r(B)^*X = \bigcup_{x:[x]_{B_r} \cap X} [x]_{B_r} \neq \emptyset$, upper approximation,
$Bnd_{N_r(B)}(X)$	$N_r(B)^*X \backslash N_r(B)_*X = \{x \in N_r(B)^*X \mid x \notin N_r(B)_*X\}$.

The original generalized approximation space (GAS) model [44] has recently been extended as a result of recent work on nearness of objects (see, *e.g.*, [12, 11, 23, 24, 28, 29, 45, 43]). A nearness approximation space (NAS) is a tuple

$$NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r}),$$

where the approximation space NAS is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbourhoods) $N_r(B)$, and neighborhood overlap function ν_{N_r}. The relation $\sim_{B_r}$ is the usual indiscernibility relation from rough set theory restricted to a subset $B_r \subseteq B$. The subscript r denotes the cardinality of the restricted subset B_r, where we consider $\binom{|B|}{r}$, *i.e.*, $|B|$ functions $\phi_i \in \mathcal{F}$ taken r at a time to define the relation $\sim_{B_r}$. This relation defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where

$$[x]_{B_r} = \{x' \in \mathcal{O} \mid \ x \sim_{B_r} x'\}.$$

These classes form a new set called the quotient set $\mathcal{O}/\sim_{B_r}$, where

$$\mathcal{O}/\sim_{B_r} = \left\{ [x]_{B_r} \mid \; x \in \mathcal{O} \right\}.$$

In effect, each choice of probe functions B_r defines a partition $\xi_{\mathcal{O},B_r}$ on a set of objects $\mathcal{O}$, namely,

$$\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r} .$$

Every choice of the set B_r leads to a new partition of $\mathcal{O}$. Let $\mathcal{F}$ denote a set of features for objects in a set X, where each $\phi_i \in \mathcal{F}$ that maps X to some *value set* V_{ϕ_i} (range of ϕ_i). The value of $\phi_i(x)$ is a measurement associated with a feature of an object $x \in X$. The overlap function ν_{N_r} is defined by

$$\nu_{N_r} : \mathcal{P}(\mathcal{O}) \times \mathcal{P}(\mathcal{O}) \longrightarrow [0, 1],$$

where $\mathcal{P}(\mathcal{O})$ is the powerset of $\mathcal{O}$. The overlap function ν_{N_r} maps a pair of sets to a number in $[0, 1]$ representing the degree of overlap between sets of objects with features defined by probe functions $B_r \subseteq B$[46]. For each subset $B_r \subseteq B$ of probe functions, define the binary relation $\sim_{B_r} = \{(x,\ x') \in \mathcal{O} \times \mathcal{O} : \forall \phi_i \in B_r, \phi_i(x) = \phi_i(x')\}$. Since each $\sim_{B_r}$ is, in fact, the usual indiscernibility relation [21], for $B_r \subseteq B$ and $x \in \mathcal{O}$, let $[x]_B$ denote the equivalence class containing x, *i.e.*,

$$[x]_{B_r} = \{x' \in \mathcal{O} \mid \; \forall f \in B_r, f(x') = f(x)\}.$$

If $(x,\ x') \in \sim_{B_r}$ (also written $x \sim_{B_r} x'$), then x and x' are said to be *B-indiscernible* with respect to all feature probe functions in B_r. Then define a collection of partitions $N_r(B)$ (families of neighborhoods), where

$$N_r(B) = \{\xi_{\mathcal{O},B_r} \mid \; B_r \subseteq B\}.$$

Families of neighborhoods are constructed for each combination of probe functions in B using $\binom{|B|}{r}$, *i.e.*, $|B|$ probe functions taken r at a time.

The family of neighbourhoods $N_r(B)$ contains a set of percepts. A *percept* is a byproduct of perception, *i.e.*, something that has been observed [15]. For example, a class in $N_r(B)$ represents *what has been perceived about objects belonging to neighbourhoods, i.e.*, observed objects with matching probe function values.

Theorem 5.1 Families of Neighbourhoods Theorem
A collection of partitions (families of neighbourhoods) $N_r(B)$ is a near set.

Proof 5.1.1 *Given a collection of partitions (families of neighbourhoods) $N_r(B)$. A partition $\xi_{\mathcal{O},B_r} \in N_r(B)$ consists of classes $[x]_{B_r}$. From Theorem 3.11, classes are near sets. Hence, from Theorem 3.16, $\xi_{\mathcal{O},B_r}$ is a near set. Hence, $N_r(B)$ is a near set.* □

A sample $X \subseteq \mathcal{O}$ can be approximated relative $B \subseteq \mathcal{F}$ by constructing a *collection of partitions* $N_r(B)$*-lower approximation* $N_r(B)_*X$, where

$$N_r(B)_*X = \bigcup_{x:[x]_{B_r} \subseteq X} [x]_{B_r},$$

and a *collection of partitions* $N_r(B)$*-upper approximation* $N_r(B)^*X$, where

$$N_r(B)^*X = \bigcup_{x:[x]_{B_r} \cap X \neq \emptyset} [x]_{B_r}.$$

Theorem 5.2 *A lower approximation $N_r(B)_*X$ of a set X is a near set.*

Proof 5.2.1 *Given a lower approximation $N_r(B)_*X$ of a set X. By definition, $N_r(B)_*X \subseteq X$ and $N_r(B)_*X$ consists of classes that are subsets of X. $N_r(B)_*X$ consists of classes $[x]_{B_r}$. From Theorem 3.11, classes are near sets. Hence, from Theorem 3.16, $N_r(B)_*X$ is a near set.* □

Theorem 5.3 *An upper approximation $N_r(B)^*X$ of a set X is a near set.*

Then $N_r(B)_*X \subseteq N_r(B)^*X$ and the boundary region $Bnd_{N_r(B)}(X)$ between upper and lower approximations of a set X is defined using set difference, *i.e.*,

$$Bnd_{N_r(B)}(X) = N_r(B)^*X \backslash N_r(B)_*X = \{x \in N_r(B)^*X \mid x \notin N_r(B)_*X\}.$$

Observation 5.4 Near Set with an Empty Boundary
It should also be observed that whenever $Bnd_{N_r(B)}(X) = \emptyset$, this means that $\left|Bnd_{N_r(B)}(X)\right| = 0$, $N_r(B)_*X = N_r(B)^*X$ and $N_r(B)_*X \subseteq X$. From this, we know that $N_r(B)_*X$ and X share objects that have matching descriptions, i.e., *objects in each class in $N_r(B)_*X$ are also objects contained in X. Recall from Theorem 3.11, also, that every class is a near set. By definition, all classes in $N_r(B)_*X$ are also subsets of X. Then it follows that X is a near set (see Theorem 5.5, case 2).*

Theorem 5.5 Neighbourhoods Approximation Boundary
A set X with an approximation boundary $\left|Bnd_{N_r(B)}(X)\right| \geq 0$ is a near set.

Proof 5.5.1 *The proof is similar to the one given for Theorem 4.8. There are two cases to consider.*

1. *$\left|Bnd_{N_r(B)}(X)\right| > 0$. Given an approximation boundary $\left|Bnd_{N_r(B)}(X)\right| > 0$ for a set X. Then $N_r(B)_*X \subset N_r(B)^*X$,* i.e., *the lower approximation $N_r(B)_*X$ is a non-empty subset of the upper approximation $N_r(B)^*X$ and $N_r(B)_*X$ is also a subset of X. Hence, from Theorem 4.8, X is a near set.*

2. $\left|Bnd_{N_r(B)}(X)\right| = 0$. *Given* $\left|Bnd_{N_r(B)}(X)\right| = 0$ *for a set* X. *Then* $N_r(B)_*X = N_r(B)^*X$ *and* $N_r(B)_*X$ *is also a subset of* X. *Also from Obs. 5.4 and Theorem 3.16,* X *is a near set.*

□

From Theorem 5.5, set X is termed a *near set* relative to a chosen collection of partitions $N_r(B)$ *iff* $|Bnd_{N_r(B)}(X)| \geq 0$. In the case where $\left|Bnd_{N_r(B)}(X)\right| > 0$, the set X has been *roughly* approximated, *i.e.*, X is a rough set as well as a near set. In the case where $\left|Bnd_{N_r(B)}(X)\right| = 0$, the set X is considered a near set but not a rough set. In effect, every rough set is a near set but not every near set is a rough set.

5.1 Sample Families of Neighbourhoods

Let $X \subseteq \mathcal{O}$, $B \subseteq \mathcal{F}$ denote a set of sample objects $\{x_0, x_1, \ldots, x_7\}$ and set of functions $\{s, a, p, r\}$, respectively. Sample values of the state function $s : X \longrightarrow \{0, 1\}$ and action function $a : X \longrightarrow \{1, 2, 3\}$ are shown in Table 2. Assume reward function $r : A \longrightarrow [0, 1]$ and a preference function $p : S \times A \longrightarrow [0, 1]$. From Table 2, we can, for example, extract the family of neighbourhoods $N_1(B)$ for $r = 1$.

$$\begin{aligned}
X &= \{x_0, x_1, \ldots, x_7\}, \\
N_1(B) &= \left\{\xi_{\{s\}}, \xi_{\{a\}}, \xi_{\{p\}}, \xi_{\{r\}}\right\}, \text{where} \\
\xi_{\{s\}} &= \left\{[x_0]_{\{s\}}, [x_2]_{\{s\}}\right\}, \\
\xi_{\{a\}} &= \left\{[x_0]_{\{a\}}, [x_1]_{\{a\}}, [x_3]_{\{a\}}\right\}, \\
\xi_{\{p\}} &= \left\{[x_0]_{\{p\}}, [x_2]_{\{p\}}, [x_5]_{\{p\}}, [x_6]_{\{p\}}\right\}, \\
\xi_{\{r\}} &= \left\{[x_0]_{\{r\}}, [x_2]_{\{r\}}\right\}.
\end{aligned}$$

5.2 Feature Selection Method

A practical outcome of the family of neighbourhoods near set approach is a feature selection method. Recall that each partition $\xi_{\mathcal{O},B_r} \in N_r(B)$ contains classes defined by the relation $\sim_{B_r}$. We are interested in the classes in each $\xi_{\mathcal{O},B_r} \in N_r(B)$ with information content greater than or equal to some threshold th. The basic idea here is to identify probe functions that lead to partitions with the highest information content, which occurs in partitions with high numbers of classes. In effect, as the number of classes in a partition

Algorithm 1: Partition Selection

Input : $NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$, $B \in \mathcal{F}$, choice r.
Output: Partition size list Φ, where $\Phi[i]$ = number of classes in partition $\xi_{\mathcal{O},B_r} \in N_r(B)$.
Initialize $i = 0$;
while $(i \leq |N_r(B)|)$ **do**
  Select i^{th} partition $\xi_{\mathcal{O},B_r} \in N_r(B)$;
  $\Phi[i] = |\xi_{\mathcal{O},B_r} \in N_r(B)|$;
  $i = i + 1$;
end

increases, there is a corresponding increase in the information content of the partition. A list Φ of partition sizes is constructed using Alg. 1.

By sorting Φ based on information content using Alg. 2, we have a means of means of selecting tuples containing probe functions that define partitions having the highest information content. To achieve feature selection with polynomial time complexity, features are selected by considering only the partitions in $N_1(B)$ (see [30] for details).

Algorithm 2: Feature Selection

Input : Array Φ, where $\Phi[i]$ = number of classes in $\xi_{\mathcal{O},B_r} \in N_r(B)$, threshold th.
Output: Ordered list Γ, where $\Gamma[i]$ is a winning probe function.
Initialize $i = 0$;
Sort Φ in descending order based on the information content of $\xi_{\mathcal{O},B_r} \in N_r(B)$;
while $(i \geq th)$ **do**
  $\Gamma[i] = \Phi[i]$;
  $i = i + 1$;
end

5.3 Overlap Function

It is now possible to formulate a basis for measuring average the degree of overlap between between near sets. Let X, Y defined in terms of a family of neighbourhoods $N_r(B)$. There are two forms of the overlap function.

$$\nu_{N_r(B)}(X,\ Y) = \begin{cases} \frac{|X \cap Y|}{|Y|}, & if\,|Y| \neq \emptyset, \\ 1, & otherwise. \end{cases} \tag{5}$$

$$\nu_{N_r(B)}(X,\ Y) = \begin{cases} \frac{|X \cap Y|}{|X|}, & if\, |X| \neq \emptyset, \\ 1, & otherwise. \end{cases} \quad (6)$$

Coverage $\nu_{N_r(B)}(X,\ Y)$ in (5) is used in cases where it is known that $|X| \leq |Y|$ (see, *e.g.*, [36, 37]). For example, coverage can be used to measure the degree that a class $[x]_{B_r}$ is covered by the lower approximation $N_r(B)_*X$ in

$$\nu_{N_r(B)}([x]_{B_r}, N_r(B)_*X) = \frac{|\,[x]_{B_r} \cap N_r(B)_*X|}{|N_r(B)_*X|},$$

called *lower rough coverage* [37]).

6 Conclusion

This article introduces a general theory of nearness of objects in a near set approach to set approximation. Near sets represent a generalization of the approach to the classification of objects introduced by Zdzisław Pawlak during the early 1980s. Near sets and rough sets are very much like two sides of the same coin. From a rough set point-of-view, the focus is on the approximation of sets with non-empty boundaries. By contrast, in a near set approach to set approximation, the focus is on the discovery of near sets in the case where there is either a non-empty or an empty approximation boundary. There are a number of practical outcomes of the near set approach, *e.g.*, feature selection [30], object recognition in images [11, 24], image processing [3], granular computing [25, 42] and in various forms of machine learning [14, 25, 31, 32, 35, 34, 37, 38]. Future work will focus on the near set approach in pattern recognition and image processing.

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References

[1] M. Banerjee, M.K. Chakraborty, Algebras from rough sets. In: Pal, S.K., Polkowski, L., Skowron, A. (Eds.): *Rough-neuro Computing: Techniques for Computing with Words*, Springer, Berlin, 2004, 157-184.

[2] M. Banerjee, M.K. Chakraborty, Rough sets through algebraic logic, *Fundamenta Informaticae*, **28** (3,4) (1996), 211-221.

[3] M. Borkowski, J.F. Peters, Matching 2D image segments with genetic algorithms and approximation spaces, *Transactions on Rough Sets*, **V**, LNCS 4100 (2006), 63-101.

[4] G. Cattaneo, D. Ciucci, Algebraic structures for rough sets, *Transactions on Rough Sets*, **II**, LNCS 3135 (2004), 63-101.

[5] D. Bianucci, G. Cattaneo, D. Ciucci, Entropies and co-entropies of coverings with application to incomplete information systems, *Fundamenta Informaticae*, **75** (2007), 77-105.

[6] M.K. Chakraborty, M. Banerjee, Rough dialogue and implication lattices, *Fundamenta Informaticae*, **75** (2007), 123-139.

[7] I. Düntsch, A logic for rough sets, *Theoretical Computer Science*, **179** (1997), 427-436.

[8] S.Z. Der, R. Chellappa, *Probe based recognition of targets in infrared images*, Research Report CAR-TR-693, Center for Automation Research, 1993.

[9] V.A. Efremovič, Geometry of proximities 1, *Mat. Sb.*, **31** (73) (1952), 189-200.

[10] P.M. Glover, M.M. Castano-Briones, A. Bassett, Z. Pikramenou, Ligand design for luminescent lanthanide complexes: From DNA recognition to sensing, *Chem. Listy*, **98** (2004), s1-s120.

[11] C. Henry, J.F. Peters, Image Pattern Recognition Using Approximation Spaces and Near Sets, In: *Proceedings of Eleventh International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing* (RSFDGrC 2007), *Joint Rough Set Symposium* (JRS 2007), *Lecture Notes in Artificial Intelligence*, **4482** (2007), 475-482.

[12] H. Herrlich, A concept of nearness, *General Topology and Applications*, **4** (1974), 191-212.

[13] E. Konrad, E. Orłowska, Z. Pawlak, *Knowledge Representation Systems*, Institute for Computer Science, Polish Academy of Sciences, Report 433, 1981.

[14] D. Lockery, J.F. Peters, Robotic target tracking with approximation space-based feedback during reinforcement learning, Springer Best Paper Award, In: *Proceedings of Eleventh International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing* (RSFD-GrC 2007), *Joint Rough Set Symposium* (JRS 2007), *Lecture Notes in Artificial Intelligence*, **4482**, 2007, 483-490.

[15] J.A. Murray, H. Bradley, W. Craigie, C. Onions, *The Oxford English Dictionary*, Oxford University Press, London, 1933.

[16] E. Orłowska, *Semantics of Vague Concepts. Applications of Rough Sets*, Institute for Computer Science, Polish Academy of Sciences, Report 469, 1982. See, also, E. Orłowska, *Semantics of Vague Concepts*, In: G.Dorn, P. Weingartner (Eds.), *Foundations of Logic and Linguistics. Problems and Solutions*, Plenum Press, London/NY, 1985, 465-482.

[17] E. Orłowska, Z. Pawlak, *Expressive Power of Knowledge Representation Systems*, Institute for Computer Science, Polish Academy of Sciences, Report 432, 1981.

[18] E. Orłowska, Verisimilitude based on concept analysis, *Studia Logica*, **XLIX**, 3 (1990), 307-320.

[19] M. Pavel, *Fundamentals of Pattern Recognition*, $2^n d$ Ed., Marcel Dekker, Inc., NY, 1993.

[20] Z. Pawlak, A. Skowron, Rudiments of rough sets, *Information Sciences*, **177** (2007), 3-27.

[21] Z. Pawlak, *Classification of Objects by Means of Attributes*, Institute for Computer Science, Polish Academy of Sciences, Report 429, 1981.

[22] Z. Pawlak, Rough sets, *International J. Comp. Inform. Science*, **11** (1982), 341-356.

[23] J.F. Peters, Near sets. Special theory about nearness of objects, *Fundamenta Informaticae*, **75**, 1-4 (2007), 407-433.

[24] J.F. Peters, Near Sets. Toward Approximation Space-Based Object Recognition. In: Yao, Y., Lingras, P., Wu, W.-Z, Szczuka, M., Cercone, N., Ślęzak, D., Eds., *Proc. of the Second Int. Conf. on Rough Sets and Knowledge Technology* (RSKT07), *Joint Rough Set Symposium* (JRS07), *Lecture Notes in Artificial Intelligence*, **4481**, Springer, Berlin (2007), 22-33.

[25] J.F. Peters, Perceptual granulation in ethology-based reinforcement learning. In: Pedrycz, W., Skowron, A., Kreinovich, V. (Eds.), *Handbook on Granular Computing*, Wiley, NY, 2007.

[26] J.F. Peters, Classification of objects by means of features, In: *Proc. IEEE Symposium Series on Foundations of Computational Intelligence* (IEEE SCCI 2007), Honolulu, Hawaii (2007), 1-8.

[27] J.F. Peters, C. Henry, D.S. Gunderson, Biologically-inspired approximate adaptive learning control strategies: A rough set approach, *International Journal of Hybrid Intelligent Systems* (2007), *in press.*

[28] J.F. Peters, A. Skowron, J. Stepaniuk, Nearness in approximation spaces. In: G. Lindemann, H. Schlilngloff et al. (Eds.), *Proc. Concurrency, Specification & Programming* (CS&P'2006). Informatik-Berichte Nr. 206, Humboldt-Universität zu Berlin (2006), 434-445.

[29] J.F. Peters, A. Skowron. Stepaniuk, J.: Nearness of objects: Extension of approximation space model, *Fundamenta Informaticae*, **79**, (2007), 1-16.

[30] J.F. Peters, S. Ramanna, Feature Selection: Near Set Approach. In: Z.W. Ras, S. Tsumoto, D.A. Zighed (Eds.), 3^{rd} *Int. Workshop on Mining Complex Data* (MCD'08), ECML/PKDD-2007, LNAI, Springer (2007), *in press.*

[31] J.F. Peters, Toward approximate adaptive learning. In: Int. Conf. Rough Sets and Emerging Intelligent Systems Paradigms in Memoriam Zdzisław Pawlak, *Lecture Notes in Artificial Intelligence*, **4585**, Springer, Berlin Heidelberg (2007), 57-68.

[32] J.F. Peters, Granular computing in approximate adaptive learning, *International Journal of Information Technology and Intelligent Computing* (2007), *in press.*

[33] J.F. Peters, C. Henry, Approximation spaces in off-policy Monte Carlo learning, *Engineering Applications of Artificial Intelligence*, **20** (5) (2007), 667-675.

[34] J.F. Peters, M. Borkowski, C. Henry, D. Lockery, D.S. Gunderson, Line-Crawling Bots that Inspect Electric Power Transmission Line Equipment. In: *Proc. Third Int. Conference on Autonomous Robots and Agents* (ICARA 2006), Palmerston North, New Zealand (2006), 39-44.

[35] J.F. Peters, M. Borkowski, C. Henry, D. Lockery, Monocular vision system that learns with approximation spaces. In: Ella, A., Lingras, P., Slezak,

D., Suraj, Z. (Eds.), *Rough Set Computing: Toward Perception Based Computing*, Idea Group Publishing, Hershey, PA (2006), 1-22.

[36] J.F. Peters, Rough ethology: Towards a Biologically-Inspired Study of Collective behaviour in Intelligent Systems with Approximation Spaces, *Transactions on Rough Sets*, **III**, LNCS 3400 (2005), 153-174.

[37] J.F. Peters, C. Henry,Reinforcement learning with approximation spaces, *Fundamenta Informaticae*, **71**, nos. 2-3 (2006), 323-349.

[38] J.F. Peters, S. Shahfar, S. Ramanna, T. Szturm, Biologically-inspired adaptive learning: A near set approach, In: *Proc. Frontiers in the Convergence of Bioscience and Information Technologies* (FBIT07), IEEE, NJ, 11 October 2007, *in press*.

[39] J.F. Peters, C. Henry, S. Ramanna, Rough Ethograms: Study of Intelligent System behaviour. In: M.A. Kłopotek, S. Wierzchoń, K. Trojanowski (Eds.), *New Trends in Intelligent Information Processing and Web Mining* (IIS05), Gdańsk, Poland (2005), 117-126.

[40] L. Polkowski, *Rough Sets. Mathematical Foundations*, Springer-Verlag, Heidelberg, 2002.

[41] J.J. Rissanen, A universal prior for integers and estimation by Minimum Description Length, *Annals of Statistics* **11**(2) (1983), 416-431.

[42] A. Skowron, J.F. Peters, Rough granular computing. In: Pedrycz, W., Skowron, A., Kreinovich, V. (Eds.), *Handbook on Granular Computing*, Wiley, NY, 2007.

[43] A. Skowron, J. Stepaniuk, J.F. Peters, R. Swiniarski, Calculi of approximation spaces, *Fundamenta Informaticae*, **72**(1-3) (2006), 363-378.

[44] A. Skowron, J. Stepaniuk, Generalized approximation spaces. In: Lin, T.Y.,Wildberger, A.M. (Eds.), *Soft Computing*, Simulation Councils, San Diego (1995), 18-21.

[45] A. Skowron, R. Swiniarski, P. Synak, Approximation spaces and information granulation, *Transactions on Rough Sets*, **III** (2005), 175-189.

[46] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, *Fundamenta Informaticae* **27**(2-3) (1996), 245-253.

[47] Tagore, R., Einstein, A.: Rabindranath Tagore's conversation with Albert Einstein. The Bangla Reporter, 11 May (2007) 35.

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