MATH 1210 Supplementary Notes: Some Geometrical Considerations in in Two- and Three-Dimensions

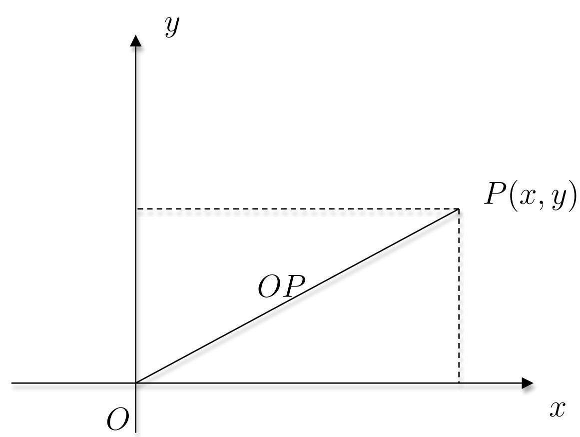
T. G. Berry and M. Davidson

Department of Mathematics, University of Manitoba

1 *Cartesian Coordinates in Two- and Three-Dimensions*

To define a two-dimensional Cartesian coordinate system:

(a) **Choose two mutually perpendicular lines.**
Each of these two lines is said to be a coordinate axis, and together are known as the coordinates axes. The point of intersection of these two lines is known as the origin of the coordinate system, and is usually labeled O.
The orientation of these two lines may be arbitrarily chosen, subject to the restriction that they must be mutually perpendicular. In most instances, one of them is chosen to be horizontal, with the other therefore being vertical. Typically the horizontal axis is known as the x-axis and the vertical axis is known as the y-axis, as shown below.



(b) **Along each coordinate axis choose a unit of measurement as measured from the origin, and a direction** (known as the"positive" direction for that axis) **in which to measure this unit for that axis.**

These units of measurement along the coordinate axes need not be the same, but are often chosen to be so.

(c) **Define the coordinates** (x, y) **of any point** P **in the plane spanned by the two coordinate axes as follows:** the x-coordinate is defined a the **directed length** of the projection of the line segment OP onto the x-axis, while the y-coordinate represents the **directed length** of the **(perpendicular or orthogonal) projection** of the line segment OP onto the y-axis.

In each case, the directed distance is positive if it is measured in the same direction as the for that axis and negative if in the opposite direction.

***Remark* 1.1.** *The coordinate axes subdivide the plane into four quadrants, depending on the signs of x and y, as follows:*

(i) first quadrant: both x and y are positive,

(ii) second quadrant: $x < 0$ and $y > 0$,

(iii) third quadrant: $x < 0$ and $y < 0$,

(iv) fourth quadrant $x > 0$ and $y < 0$.

***Remark* 1.2.** *The origin has coordinates $(0, 0)$, while any point on the x-axis has coordinates $(x, 0)$ and any point on the y-axis has coordinates $(0, y)$.*

***Remark* 1.3.** *We note that the length of the line segment OP is given by the simple equation*

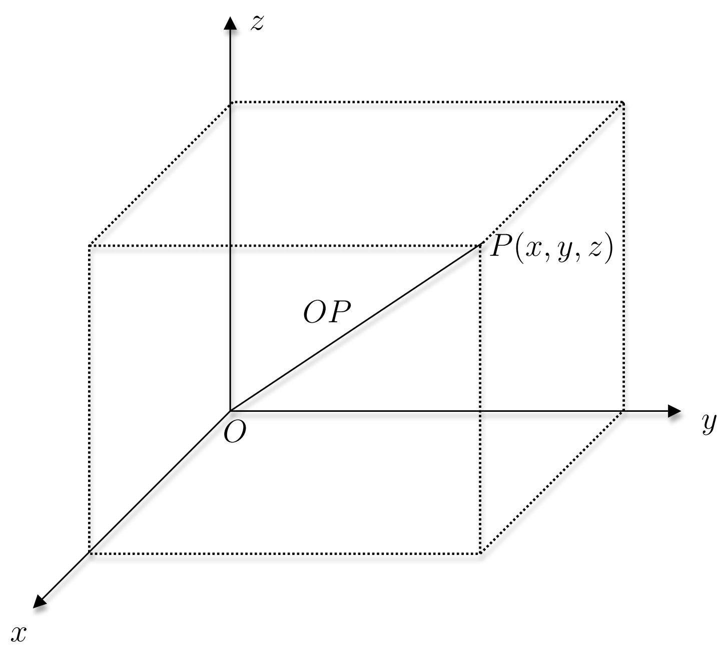
$$|OP| = \sqrt{x^2 + y^2}, \tag{1.1}$$

which is effectively a restatement of the Pythagorean Theorem.
More generally, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points in the plane, the length of the line segment P_1P_2 is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1.2}$$

***Remark* 1.4.** *The 2-dimensional plane endowed with Cartesian coordinates (x, y) is known as the* **2-dimensional Euclidean Plane** *and is denoted by $\mathbb{E}^2$.*

To define a three-dimensional Cartesian coordinate system, we follow a similar procedure but begin with **three mutually perpendicular coordinate axes**, which intersect at a common origin, choosing a unit of measurement and positive direction along each. Typically these axes are referred to as the x-axis, y-axis, and z-axis respectively. In this case the **coordinates** (x, y, z) of any point P in the three-dimensional space spanned by these three coordinate axes, are defined once again as the **directed length** of the projection of the line segment OP onto the corresponding coordinate axis, subject to the **convention** that this directed distance is positive if measured in the same direction as the corresponding unit, and negative if measured in the opposite direction.



***Remark* 1.5.** *In 3-dimensions the length of the line segment OP is given by*

$$|OP| = \sqrt{x^2 + y^2 + z^2}. \tag{1.3}$$

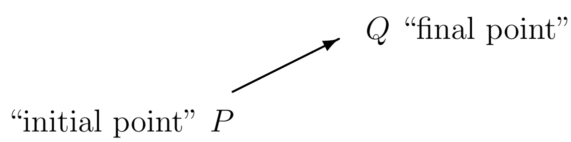
More generally, if $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points in space, the length of the line segment P_1P_2 is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1.4}$$

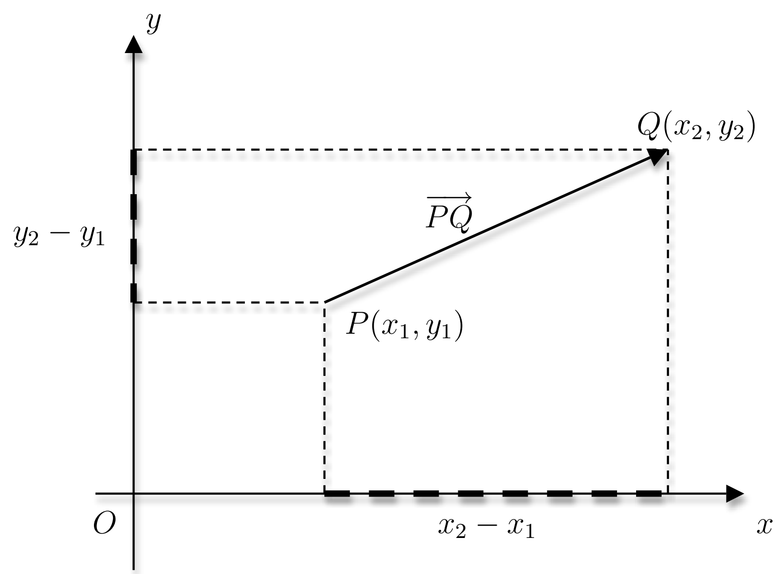
***Remark* 1.6.** *Three-dimensional space endowed with Cartesian coordinates (x, y, z) is known as* **Euclidean 3-dimensional space** *and is denoted by $\mathbb{E}^3$.*

2 *Vectors in* $\mathbb{E}^2$ *and* $\mathbb{E}^3$

A **vector** is a geometric object which has both **magnitude** and **direction**. Geometrically, a vector is represented as a directed line segment (or "arrow"). Typically a vector is specified through the use two points P and Q, with an assigned direction from the ***"initial point"*** (also called the ***"point of application"*** or ***"point of attachment"*** of the vector), to the ***"final point"*** . For example , a two-dimensional vector $\overrightarrow{PQ}$ (lying in the plane of this page) may be represented as



In order to turn a two-dimensional geometric vector into an algebraic vector , we now superimpose a two-dimensional Cartesian coordinate system onto this plane, as in:



In addition, if the coordinates of P are (x_1, y_1) and the coordinates of Q are (x_2, y_2), the quantities $(x_2 - x_1)$ and $(y_2 - y_1)$ respectively represent the **directed lengths** of the projection of the vector $\overrightarrow{PQ}$ onto the x-axis and y-axis respectively.
These quantities are known respectively as the *"x-component"* and *"y-component"* of the vector $\overrightarrow{PQ}$, and we write $\overrightarrow{PQ}$ in terms of its components as

$$\overrightarrow{PQ} = [(x_2 - x_1), (y_2 - y_1)] \tag{2.1}$$
$$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} \tag{2.2}$$

in which

$\hat{\imath} = [1, 0]$ represents the *"unit vector in the x-direction"*
and
$\hat{\jmath} = [0, 1]$ represents the *"unit vector in the y-direction"*.

***Remark* 2.1.** *The pair of vectors $\hat{\imath}$ and $\hat{\jmath}$ are said to the "unit basis vectors" for the Cartesian coordinate system $\mathbb{E}^2$.*

***Remark* 2.2.** *In the above we have introduced two alternate notations for a vector, namely*

(i) in terms of its components, viz., $[(x_2 - x_1), (y_2 - y_1)]$,

*in which square brackets are used to **distinguish the vector from a point**,*

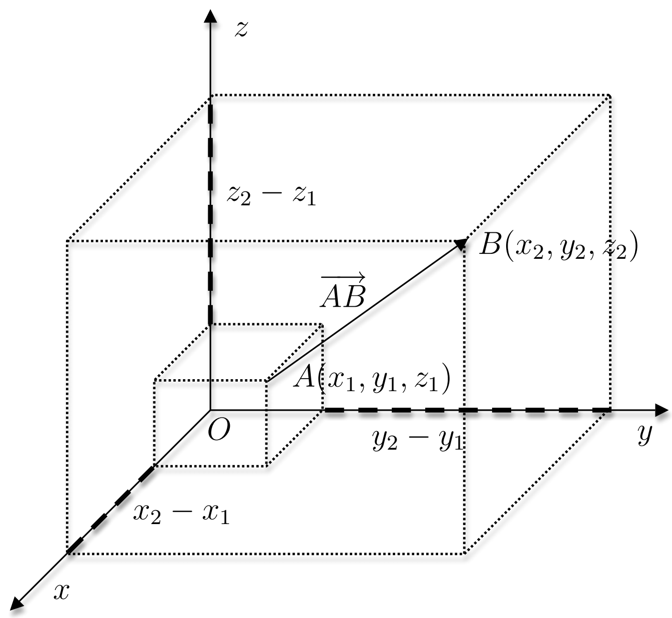
(ii) in terms of its components and the "unit basis vectors", *viz., $(x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$.*

***Remark* 2.3.** *Although either of the above notations for a vector is acceptable, **the second is preferred as it does not permit the possibility of confusion as to whether we are considering a vector or a point** (in spite of the caution at the end of the first part of the above remark) .*

***Remark* 2.4.** *The length (also known as the "norm") of the vector $\overrightarrow{PQ}$, denoted by $\|\overrightarrow{PQ}\|$ is simply the length of the line segment PQ, i.e.,*

$$\|\overrightarrow{PQ}\| = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{2.3}$$

Vectors in space (also known as three-dimensional vectors) are similar (again being determined by two points), although they have three components. Indeed the x- , y- and z-components of a 3-d vector are determined in a completely analogous fashion to that introduced above, as illustrated in the following diagram:



***Remark* 2.5.** *By analogy with the preceding case, a 3-d vector $\overrightarrow{AB}$ may be written in either of the following two forms:*

$$\overrightarrow{AB} = [(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)] \tag{2.4}$$

$$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k} \tag{2.5}$$

in which

$\hat{\imath} = [1, 0, 0]$ *represents the* "unit (basis)vector in the x-direction",
$\hat{\jmath} = [0, 1, 0]$ *represents the* "unit (basis) vector in the y-direction"
and
$\hat{k} = [0, 0, 1]$ *represents the* "unit (basis)vector in the z-direction",

and (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively denote the coordinates of the two points A and B.

***Remark* 2.6.** *The "norm" of the 3-d vector $\overrightarrow{AB}$, denoted by $\|\overrightarrow{AB}\|$ is again simply the length of the line segment AB, i.e.,*

$$\|\overrightarrow{AB}\| = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{2.6}$$

3 *Algebraic Operations on Vectors in* $\mathbb{E}^2$ *and* $\mathbb{E}^3$

Throughout this section we will assume that all vectors are either two- or three-dimensional, but ***for expediency will work only with the three-dimensional case***. The case of 2-d vectors may be easily extracted by assuming that the third component is always 'zero'. ***All results are valid in both cases, with the exception of the "vector product" (or "cross product") of two vectors, which is only meaningful when the two vectors are 3-dimensional***.

The Zero Vector

Any vector having ***no magnitude*** is said to be a ***zero vector***. Of course, there are many zero vectors, but they all share a common property, namely, all of their components (in any number of dimensions) are "zero". In this sense the "zero vector" is unique, and is therefore denoted by $\overrightarrow{0}$ independent of the dimensionality of the space; however, it is implicitly assumed that in a given context the number of components of the zero vector is consistent with the dimensionality of the space in which we are currently working.

Equality of Vectors

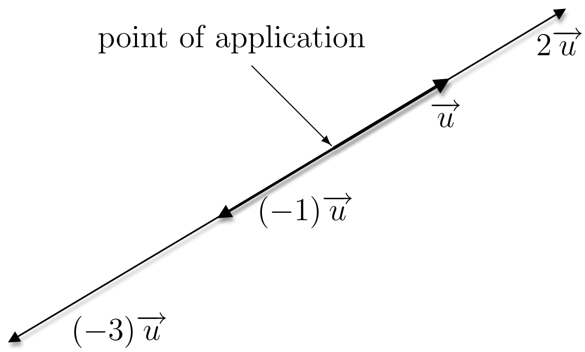
Two vectors are said to be ***equal*** if they have the same magniutude and direction.
Clearly, two vectors are equal if and only if their corresponding components are the same, i.e., if $\overrightarrow{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then

$$\overrightarrow{u} = \overrightarrow{v} \iff (u_1 = v_1 \quad and \quad u_2 = v_2 \quad and \quad u_3 = v_3). \tag{3.1}$$

***Remark* 3.1.** *Since the point of application of the vectors is not mentioned in the above definition, we may effectively move a vector from one point to another preserving its fundamental geometric properties of length and direction. Even though the vector has been moved to another point, it is regarded as being the same vector. This operation is known as the "Euclidean notion of parallelism", and we emphasize that it preserves both the length and direction of a given vector.*
As a consequence, if $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ ***is a vector attached to a point*** P ***with coordinates*** (x_0, y_0, z_0) ***then we may also identify*** $\overrightarrow{v}$ ***as being the vector attached to O having the point*** (v_1, v_2, v_3) ***as its final point****, i.e.,* $\overrightarrow{v} = \overrightarrow{O(v_1, v_2, v_3)}$.

Scalar Multiplication of a Vector

Suppose that λ is a given real number and that $\overrightarrow{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ is a given vector. The vector obtained by scalar multiplication of $\overrightarrow{u}$ by λ, denoted by $\lambda\overrightarrow{u}$, is the vector obtained by "scaling" $\overrightarrow{u}$ by the factor λ, where the "scaling" involves expansion if $|\lambda| > 1$, contraction if $0 < |\lambda| < 1$, together with a **reversal of direction** in the case when $\lambda < 0$. We illustrate this concept in the following diagram:



***Remark* 3.2.** *In terms of components, scalar multiplication of $\overrightarrow{u}$ by λ may be written as*

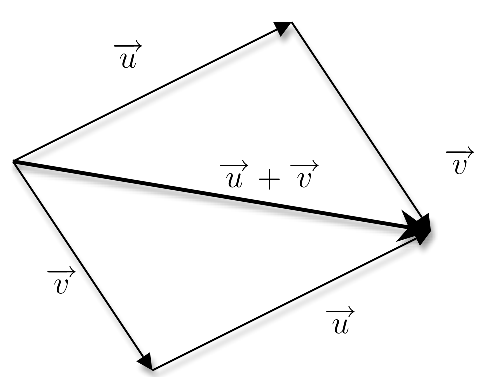
$$\lambda\overrightarrow{u} = (\lambda u_1)\hat{i} + (\lambda u_2)\hat{j} + (\lambda u_3)\hat{k} \tag{3.2}$$

***Remark* 3.3.** *Multiplication of a vector $\overrightarrow{u}$ by the scalar (-1) results in a vector denoted by $-\overrightarrow{u}$, which is simply the "reversal" (or negative) of $\overrightarrow{u}$.*

***Remark* 3.4.** *Two or more vectors that are simply scalar multiples of one another are said to be **parallel**.*

Addition of Vectors

Two vectors may be added together, using the so-called "***parallelogram law*** ", by moving one vector so that its initial point coincides with the final point of the other vector. Graphically, the sum of the given vectors $\overrightarrow{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, denoted by $\overrightarrow{u} + \overrightarrow{v}$, is the ***diagonal*** in the parallelogram having $\overrightarrow{u}$ and $\overrightarrow{v}$ as its edges, as shown below:



***Remark* 3.5.** *It is clear from the above diagram that $\overrightarrow{u}+\overrightarrow{v}$ and $\overrightarrow{v}+\overrightarrow{u}$ represent the same vector, and hence we may write*

$$\overrightarrow{u}+\overrightarrow{v}=\overrightarrow{v}+\overrightarrow{u}.$$

***Remark* 3.6.** *In terms of components, vector addition involves simply the addition of corresponding vector components, as in*

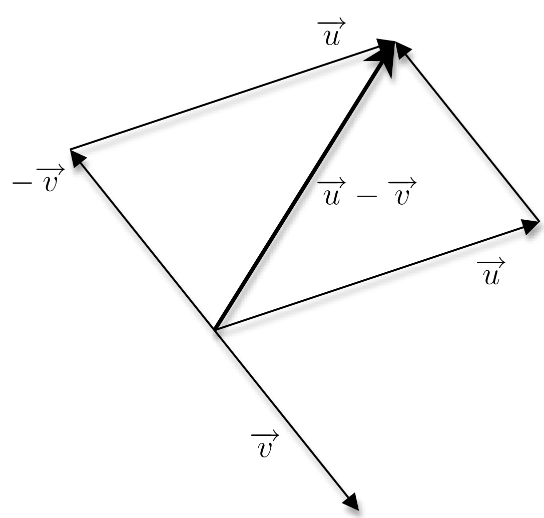
$$\begin{aligned}\overrightarrow{u}+\overrightarrow{v} &= (u_1\hat{i}+u_2\hat{j}+u_3\hat{k})+(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) \qquad (3.3)\\ &= (u_1+v_1)\hat{i}+(u_2+v_2)\hat{j}+(u_3+v_3)\hat{k}. \qquad (3.4)\end{aligned}$$

The Difference of Two Vectors (also known as "Vector Subtraction")

The difference of two vectors $\overrightarrow{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ and $\overrightarrow{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ is constructed by using two vector operations, namely, scalar multiplication and vector addition as in

$$\overrightarrow{u}-\overrightarrow{v}=\overrightarrow{u}+(-1)\overrightarrow{v} \qquad (3.5)$$

Graphically, this operation may depicted, again by the parallelogram law, being careful to use $\overrightarrow{u}$ and $-\overrightarrow{v}$ as the edges of the parallelogram, as shown below:



***Remark* 3.7.** *Unlike the case of vector addition, the difference of vectors is very crucially dependent on the order of the vectors,*

i.e., $\overrightarrow{u} - \overrightarrow{v}$ *and* $\overrightarrow{v} - \overrightarrow{u}$ *are* **NOT THE SAME**, *although it is easily shown that*

$$\overrightarrow{u} - \overrightarrow{v} = -(\overrightarrow{v} - \overrightarrow{u}) \ .$$

You may easily verify this by inserting the vector $-\overrightarrow{u}$ *on the above diagram and completing the appropriate parallelogram to determine the vector* $\overrightarrow{v} - \overrightarrow{u}$.

Remark* 3.8.** *In terms of components, the difference of two vectors involves simply the difference of their respective components* ***taken in the correct order*:*

$$\overrightarrow{u} - \overrightarrow{v} = (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) - (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \tag{3.6}$$

$$= (u_1 - v_1)\hat{i} + (u_2 - v_2)\hat{j} + (u_3 - v_3)\hat{k}. \tag{3.7}$$

Two Other Useful Operations on A Pair of Vectors

In addition to the above ***arithmetic operations*** on vectors, there are two additional ways in which two vectors may be combined, namely, through the ***dot product*** and the ***cross product***:

The Dot Product of Two Vectors

Again suppose that $\overrightarrow{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ are two given vectors and θ denotes the angle between them. The ***dot product of*** $\overrightarrow{u}$ and $\overrightarrow{v}$, denoted $\overrightarrow{u} \cdot \overrightarrow{v}$, is defined by

$$\overrightarrow{u} \cdot \overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\|cos(\theta). \tag{3.8}$$

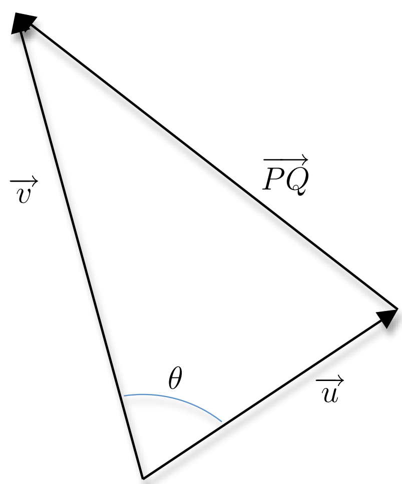
***Remark* 3.9.** *Provided that* $\overrightarrow{u} \neq \overrightarrow{0}$ *and* $\overrightarrow{v} \neq \overrightarrow{0}$, $\overrightarrow{u} \cdot \overrightarrow{v} = 0 \iff cos(\theta) = 0$

$$\iff \overrightarrow{v} \text{ and } \overrightarrow{v} \text{ are perpendicular.}$$

Remark* 3.10.** *The concept of perpendicularity mentioned in the previous remark is only useful in 2- and 3-dimensions. Although, for the purposes of this discussion, we shall restrict consideration only to* $\mathbb{E}^2$ *and* $\mathbb{E}^3$, *it is worthwhile mentioning that in higher dimensional spaces two vectors which satisfy the condition* $\overrightarrow{u} \cdot \overrightarrow{v} = 0$ *are said to be "*orthogonal****".*

Although (3.8) provides a convenient geometrical interpretation of the dot product of two vectors, this formula is not very convenient to use in the case when the vectors are expressed in terms of their components.

To derive a corresponding formula in terms of the components of the given vectors, we suppose that these vectors are positionnned at the origin (although for convenience sake the axes have been omitted from the diagram) and that $\overrightarrow{PQ}$ denotes the vector $(v_1 - u_1)\hat{i} + (v_2 - u_2)\hat{j} + (v_3 - u_3)\hat{k}$, as shown in the following diagram:



By virtue of the cosine law, we may write

$$\|\overrightarrow{PQ}\|^2 = \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 - 2\|\overrightarrow{u}\|\|\overrightarrow{v}\|cos(\theta). \tag{3.9}$$

But $\overrightarrow{PQ} = \overrightarrow{v} - \overrightarrow{u}$, so that(3.6) becomes

$$\|\overrightarrow{u}\|\|\overrightarrow{v}\|cos(\theta) = \frac{1}{2}\left(\|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 - \|(\overrightarrow{v} - \overrightarrow{u})\|^2\right) \tag{3.10}$$

or equivalently

$$\overrightarrow{u} \cdot \overrightarrow{v} = \frac{1}{2}\left(\|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 - \|(\overrightarrow{v} - \overrightarrow{u})\|^2\right). \tag{3.11}$$

Finally, we express each of the three norms appearing in the right-hand side of (3.11) in terms of the components of the corresponding vectors, and simplify the resulting algebraic expression to obtain the desired result, namely

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2 + u_3v_3. \tag{3.12}$$

Formula (3.12) **provides the desired formula for the dot product of two vectors in terms of their components**.

***Remark* 3.11.** *The above result allows us to interpret the norm of a given vector as the square root of the dot product of the vector with itself,*
i.e.,

$$\|\overrightarrow{u}\| = \sqrt{\overrightarrow{u} \cdot \overrightarrow{u}} \iff \|\overrightarrow{u}\|^2 = \overrightarrow{u} \cdot \overrightarrow{u} \tag{3.13}$$

***Remark* 3.12.** *There are a number of properties of the dot product that are useful. We state these for future reference, and recommend that you verify them for yourself:*

(i) $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}$

(ii) $c\,(\overrightarrow{u} \cdot \overrightarrow{v}) = (c\,\overrightarrow{u}) \cdot \overrightarrow{v} = \overrightarrow{u} \cdot (c\,\overrightarrow{v})$

(iii) $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}$

Cross Product of Two Vectors

Again suppose that $\overrightarrow{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ are two given vectors and θ denotes the angle between them. (The significance of the angle θ will become evident at a later time.)

The ***cross product of*** $\overrightarrow{u}$ and $\overrightarrow{v}$, denoted $\overrightarrow{u} \times \overrightarrow{v}$, is defined by

$$\overrightarrow{u} \times \overrightarrow{v} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}. \tag{3.14}$$

The cross product has some very useful properties, which we list below:

(i) if $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{u} \times \overrightarrow{v}$ are all **non-zero** vectors, then $\overrightarrow{u} \times \overrightarrow{v}$ is perpendicular to both $\overrightarrow{v}$ and $\overrightarrow{u}$, i.e.,

$\overrightarrow{u} \cdot (\overrightarrow{u} \times \overrightarrow{v}) = 0$ and $\overrightarrow{v} \cdot (\overrightarrow{u} \times \overrightarrow{v}) = 0$

as may easily be verified by direct computation. This is left as an exercise for you to do.

We will exploit this property later when we wish to find a vector which is perpendicular to two given vectors.

(ii) The square of the norm of $\overrightarrow{u} \times \overrightarrow{v}$ may be expressed entirely in terms of the norms of the given two vectors and their dot product, by the following relation

$$\|\overrightarrow{u} \times \overrightarrow{v}\|^2 = \|\overrightarrow{u}\|^2\|\overrightarrow{v}\|^2 - (\overrightarrow{u} \cdot \overrightarrow{v})^2 \tag{3.15}$$

which is known as ***Lagrange's identity***. Again this result may be easily derived by writing the various terms in this equation in terms of the components of the vectors and performing the appropriate algebraic simplification, and is therefore left as an exercise.

(iii) Other useful properties, which will again be left as exercises, are:

$$\begin{gathered}
\overrightarrow{u} \times \overrightarrow{v} = -(\overrightarrow{v} \times \overrightarrow{u}) \\
\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} \times \overrightarrow{v}) + (\overrightarrow{u} \times \overrightarrow{w}) \\
(\overrightarrow{u} + \overrightarrow{v}) \times \overrightarrow{w} = (\overrightarrow{u} \times \overrightarrow{w}) + (\overrightarrow{v} \times \overrightarrow{w}) \\
k\,(\overrightarrow{u} \times \overrightarrow{v}) = (k\overrightarrow{u}) \times \overrightarrow{w} = \overrightarrow{u} \times (k\overrightarrow{w})\ ,\ \overrightarrow{u} \times \overrightarrow{u} = \overrightarrow{0}, \\
\overrightarrow{u} \times \overrightarrow{0} = \overrightarrow{0} \times \overrightarrow{u} = \overrightarrow{0}.
\end{gathered}$$

The cross product of two vectors has a very useful geometrical property related to the angle θ between the two vectors. To derive this property, we employ Lagrange's identity (3.15) and (3.8) to write

$$\begin{aligned}
\|\overrightarrow{u} \times \overrightarrow{v}\|^2 &= \|\overrightarrow{u}\|^2\|\overrightarrow{v}\|^2 - (\overrightarrow{u} \cdot \overrightarrow{v})^2 \\
&= \|\overrightarrow{u}\|^2\|\overrightarrow{v}\|^2 - (\|\overrightarrow{u}\|\|\overrightarrow{v}\|cos(\theta))^2 \\
&= \|\overrightarrow{u}\|^2\|\overrightarrow{v}\|^2\left(1 - cos^2(\theta)\right) \\
&= (\|\overrightarrow{u}\|\|\overrightarrow{v}\|sin(\theta))^2\,,
\end{aligned}$$

from which we deduce that

$$\|\overrightarrow{u} \times \overrightarrow{v}\| = \|\overrightarrow{u}\|\|\overrightarrow{v}\|sin(\theta) \tag{3.16}$$

since $0 \leq \theta \leq \pi$.

Note how equation (3.16) ***complements equation*** (3.8), ***and displays an important connection between the dot product and the norm of the cross product of two given vectors***.

Remark 3.13. *It is an interesting exercise to confirm that the following results are valid:*

$$\begin{gathered}
\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \overrightarrow{0}, \\
\hat{\imath} \times \hat{\jmath} = -(\hat{\jmath} \times \hat{\imath}) = \hat{k}, \\
\hat{\imath} \times \hat{k} = -(\hat{k} \times \hat{\imath}) = -\hat{\jmath}, \\
\hat{\jmath} \times \hat{k} = -(\hat{k} \times \hat{\jmath}) = \hat{\imath}.
\end{gathered}$$

4 *Lines in* $\mathbb{E}^2$ *and* $\mathbb{E}^3$

A line (in two- or three-dimensions) may be specified in a variety of ways, such as:

(i) by specifying two points on the line,

(ii) by specifying a point on the line and a vector at that point (which determines the direction of the line),

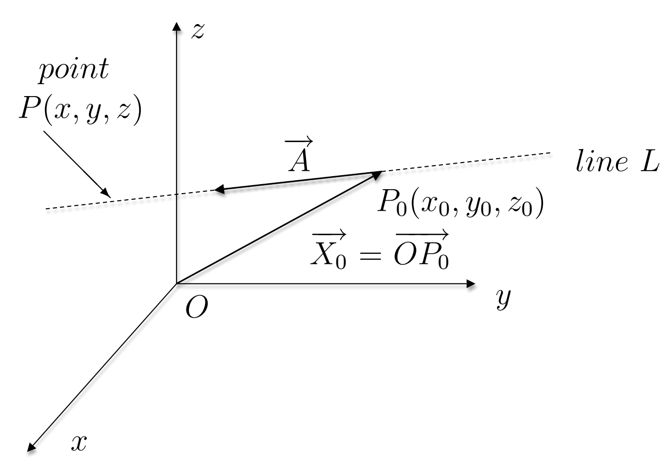
(iii) by specifying two vectors, the first of which is attached to the origin (and hence represents the position vector of its terminal point, which lies on the line) and the second of which is attached to the terminal point of the first (and which therefore represents the direction of the line).

***Remark* 4.1.** *Although the three descriptions given above are equivalent, the third is the most useful as it allows us to write down equations for the given line using only the two vector operations of scalar multiplication and vector addition introduced earlier.*

Using the above representations for a line, we may easily derive equations for a line in 2- or 3-dimensional space.

For the purposes of our discussion, we will assume that we are working in three-dimensions and make use of the standard three-dimension Cartesian Coordinate system (x, y, z). To simplify the discussion to the case of a line in a plane, we need only suppress one dimension.

Consider the case of the line L which passes through the point $P_0(x_0, y_0, z_0)$ (determined by the position vector $\overrightarrow{X_0} = \overrightarrow{OP_0} = x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}$) in the direction of the vector $\overrightarrow{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ at P_0, as shown in following diagram.



Using the operations of scalar multiplication of a vector and vector addition, we may write the position vector of any arbitrary point $P(x, y, z)$ on L as

$$\overrightarrow{OP} = \overrightarrow{X_0} + t\overrightarrow{A} \tag{4.1}$$

or in component form

$$x\hat{i} + y\hat{j} + z\hat{k} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k} + t(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \tag{4.2}$$

in which the scalar multiplier "t" appearing in this expression is known as a *parameter* along the line L, and varies as P moves along L. In particular, if $t = 0$ then P coincides with P_0, while if $t > 0$ then P lies to the side of P_0 determined by the direction vector $\overrightarrow{A}$ along L, while if $t < 0$ then P lies to the side of P_0 determined by the **negative** of the direction vector $\overrightarrow{A}$ along L. In particular, it should be noted that length of the vector $\overrightarrow{P_0P}$ is the absolute value of t times the length of $\overrightarrow{A}$
i.e.,

$$\|\overrightarrow{P_0P}\| = |t|\ \|\overrightarrow{A}\|$$

Equation (4.2) is known as the ***vector parametric equation of the line*** L.

Often one would prefer to work with the so-called ***scalar parametric equations of the line*** L, which may be obtained from (4.2) by decomposing this vector equation into component form, to obtain the three *scalar* equations

$$x = x_0 + a_1t, \quad y = y_0 + a_2t, \quad z = z_0 + a_3t. \tag{4.3}$$

***Remark* 4.2.** *One should note very carefully how the components of the vectors* $\overrightarrow{X_0}$ *and* $\overrightarrow{A}$ *enter into the scalar equations* (4.3)*. Do NOT reverse their roles; they are not interchangeable!*

Finally, we write the equations of the above line in "***symmetric form***" by eliminating the parameter "t", *if possible*, to obtain

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \tag{4.4}$$

***Remark* 4.3.** *Again, one should note very carefully how the components of the vectors* $\overrightarrow{X_0}$ *and* $\overrightarrow{A}$ *enter into the symmetric equations* (4.4)*. Once again,they are not interchangeable!*

***Remark* 4.4.** *As mentioned previously, the above development works equally well for lines in* $\mathbb{E}^2$ *and* $\mathbb{E}^3$*. If, for example, one wishes to work with a line in the xy-plane, it is only necessary to set* $z = 0$, $z_0 = 0$ *and* $a_3 = 0$ *in* (4.2) *or* (4.3) *in order to get parametric equations for the line. On the other hand, when one eliminates the parameter in this case the resulting "symmetric equation" becomes*

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} \tag{4.5}$$

which the ***standard equation*** *for a line in the xy-plane through the point* $P_0\ (x_0, y_0)$ *in the direction of the 2-d vector* $\overrightarrow{A} = a_1\hat{i} + a_2\hat{j}$

5 *Planes in* $\mathbb{E}^3$

A plane in space may be specified in a variety of ways, such as:

(i) by specifying three non-collinear points on the plane,

(ii) by specifying a point on the plane and two distinct non-parallel vectors at that point (each of which lies in the plane),

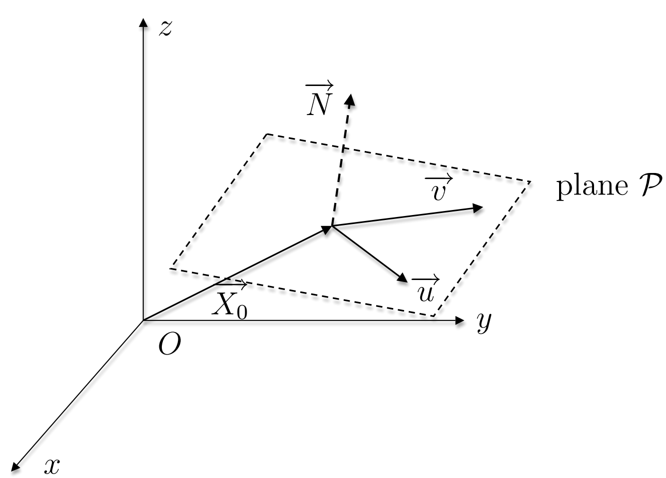
(iii) by specifying three vectors, the first of which is attached to the origin (and hence represents the position vector of its terminal point, which lies on the plane) and the other two of which are distinct and non-parallel and attached to the terminal point of the first (and which therefore each lie in the plane),

(iv) by specifying two vectors, the first of which is attached to the origin (and hence represents the position vector of its terminal point, which lies on the plane) and the second of which is attached to the terminal point of the first (and represents a vector perpendicular to the plane).

***Remark* 5.1.** *The normal vector referred to in* (iv) *above may be viewed as the cross product of the two distinct non-parallel vectors lying in the plane as mentioned in* (ii) *or* (iii) *above.*

By analogy with the case of a line, we may use the preceding representations for a plane in order to determine various forms for the equation(s) of a plane. We begin with the so-called ***parametric equations*** for a plane, which may be obtained as follows:

Consider the case of the plane $\mathcal{P}$ which passes through the point $P_0(x_0, y_0, z_0)$ (determined by the position vector $\overrightarrow{X_0} = \overrightarrow{OP_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$) and containing the two vectors $\overrightarrow{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v\hat{k}$ at P_0 as shown in following diagram.



The position vector of any arbitrary point $P(x, y, z)$ on $\mathcal{P}$ may be represented by adding appropriate scalar multiples of $\overrightarrow{u}$ and $\overrightarrow{v}$ to $\overrightarrow{X_0}$, as in

$$\overrightarrow{OP} = \overrightarrow{X_0} + s\,\overrightarrow{u} + t\,\overrightarrow{v} \tag{5.1}$$

or equivalently, in component form as

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k} + s(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}) + t(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}) \tag{5.2}$$

in which "s" and "t" denote the appropriate scalar factors, and are known as ***parameters*** for the plane $\mathcal{P}$. The above vector equation is known as the ***vector parametric equation*** for the plane $\mathcal{P}$.

Often one would prefer to work with the so-called ***scalar parametric equations of the plane*** $\mathcal{P}$, which may be obtained from (5.2) by decomposing this vector equation into component form, to obtain the three *scalar* equations

$$x = x_0 + u_1 s + v_1 t, \quad y = y_0 + u_2 s + v_2 t, \quad z = z_0 + u_3 s + v_3 t. \tag{5.3}$$

An alternate representation for the plane $\mathcal{P}$ may be obtained by using the information in (iv) above. Suppose that $\mathcal{P}$ passes through the point $P_0(x_0, y_0, z_0)$ (determined by the position vector $\overrightarrow{X_0} = \overrightarrow{OP_0} = x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}$) and is perpendicular to the specified vector $\overrightarrow{N} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$ at P_0. If $P(x, y, z)$ is any point on $\mathcal{P}$, the vector $\overrightarrow{P_0P} = (x-x_0)\hat{\imath} + (y-y_0)\hat{\jmath} + (z-z_0)\hat{k}$ lies in $\mathcal{P}$ and is therefore perpendicular to $\overrightarrow{N}$. Thus, the equation

$$\overrightarrow{N} \cdot \overrightarrow{P_0P} = 0$$

or equivalently

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \tag{5.4}$$

is the equation of the plane passing through the point $P_0(x_0, y_0, z_0)$ with "***normal vector***" $\overrightarrow{N} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$ at P_0. Equation (5.4) is said to be the "***point-normal equation***" for this plane.

Remark 5.2. *Note very carefully the roles of the vector* $\overrightarrow{N} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$ *and the point* $P_0(x_0, y_0, z_0)$ *in equation* (5.4)*. Equation* (5.4) *is often written in "**standard form**" as*

$$Ax + By + Cz = D \tag{5.5}$$

in which $[A, B, C] = \overrightarrow{N}$ *and* $D = \overrightarrow{N} \cdot \overrightarrow{OP_0}$.

Remark 5.3. *It is a relatively simple exercise to show that if one eliminates the parameters "s" and "t" from* (5.3)*, these scalar parametric equations reduce to the point-normal equation* (5.4) *for the plane* $\mathcal{P}$.