The theorem uses a symbol and function called the **factorial**, defined either explicitly by

$$0! = 1 \quad \text{and} \quad n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \quad (n \geq 1)$$

or recursively by

$$0! = 1, \quad (n+1)! = (n+1)n! \quad (n \geq 0).$$

This function is used to form integer functions of two variables called (for reasons that will become obvious) **binomial coefficients** and defined by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\cdots(n-r+1)}{r!},$$

where r is a non-negative integer and n is an integer no smaller than r. The fact that the middle expression, which is the real definition, is actually equal to the third expression is one of hundreds of facts that are known about these functions. The first fact is that they are all integers in spite of their apparently being defined as fractions. Other facts, which are simple exercises in algebra to verify, are

$$\binom{n}{0} = \binom{n}{n} = 1, \tag{1}$$

$$\binom{n}{r} = \binom{n}{n-r}, \tag{2}$$

and

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}. \tag{3}$$

The above facts can be used to prove by induction **the binomial theorem.** Where n is a positive integer,

$$(1+z)^n = \sum_{r=0}^{n} \binom{n}{r} z^r.$$

Proof. It is easy to see that

$$1 + z = \binom{1}{0} z^0 + \binom{1}{1} z^1 = \sum_{r=0}^{1} \binom{1}{r} z^r$$

and that

$$(1+z)^2 = \binom{2}{0} z^0 + \binom{2}{1} z^1 + \binom{2}{2} z^2 = \sum_{r=0}^{2} \binom{2}{r} z^r,$$

though it is not obvious at first glance. It is useful to check up on this if the symbols are new to you. The induction assumption is that k is an integer such that

$$(1+z)^k = \sum_{r=0}^{k} \binom{k}{r} z^r. \tag{4}$$

And we must prove on this basis that

$$(1+z)^{k+1} = \sum_{s=0}^{k+1} \binom{k+1}{s} z^s. \tag{5}$$

Then the theorem will be proved by the principle of mathematical induction. The verification is not difficult, but it has more easy steps than most exercises. One point that needs to be noticed is that it makes life easier to change the index of summation from (4) to (5). We begin the proof of the inductive step by noting that the left side of (5) equals

$$(1+z)(1+z)^k.$$

It is obvious how to use the induction assumption to get an equal expression,

$$(1+z)\sum_{r=0}^{k}\binom{k}{r}z^r.$$

Multiplying out gives the equal expression,

$$\sum_{r=0}^{k}\binom{k}{r}z^r+\sum_{r=0}^{k}\binom{k}{r}z^{r+1}.$$

The first of the unexpected steps, which it is easy to see the correctness of but hard at first to see the wisdom of, is to take the first term off the first sum and the last term off the second sum using fact (1) to get

$$1+\sum_{r=1}^{k}\binom{k}{r}z^r+\sum_{r=0}^{k-1}\left\{\binom{k}{r}z^{r+1}\right\}+z^{k+1}.$$

The other unexpected step is to change the index of summation in **both** sums, letting $s=r$ in the first and $s=r+1$ in the second. These changes give

$$1+\sum_{s=1}^{k}\binom{k}{s}z^s+\sum_{s=1}^{k}\left\{\binom{k}{s-1}z^s\right\}+z^{k+1}.$$

The common index and limits of summation allow the sums with sigmas to be combined to give

$$1+\sum_{s=1}^{k}\left\{\left[\binom{k}{s}+\binom{k}{s-1}\right]z^s\right\}+z^{k+1}.$$

The fact labelled (3) above allows this expression to be simplified to

$$1+\sum_{s=1}^{k}\left\{\binom{k+1}{s}z^s\right\}+z^{k+1}.$$

Then the fact (1) used twice allows the summation to be expanded to include the two other terms and equal the right side of (5),

$$\sum_{s=0}^{k+1}\binom{k+1}{s}z^s,$$

as was required. This completes the inductive proof.

If one wants the binomial theorem as it is often quoted, one just has to notice that z can be $\frac{b}{a}$ and that

$$(a+b)^n=a^n\left(1+\frac{b}{a}\right)^n=a^n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{b}{a}\right)^r=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}\,b^r.$$

This statement can be proved directly, but using $1+z$ allows a proof without both as and bs, which is slightly easier to follow.