This is the second part of a two-part document.

MATH 1210 Eigenproblem from Mechanics—Solution

The problem is transformed into an eigenvalue/eigenvector problem when the $e^{\omega t}$ (which can never be zero) has been cancelled from both sides of the following equation, obtained by substituting for $\ddot{\mathbf{y}}$ and $\mathbf{y}$ in (1):

$$\omega^2 e^{\omega t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{bmatrix} e^{\omega t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

Since the point of this example is to show how this works, we shall solve the problem for values of the constants that make the arithmetic easy: $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$. The determinant in the first stage of the solution is then

$$\begin{vmatrix} -\frac{k_1+k_2}{m_1} - \omega^2 & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} - \omega^2 \end{vmatrix} = \begin{vmatrix} -\frac{3+2}{1} - \omega^2 & \frac{2}{1} \\ \frac{2}{1} & -\frac{2}{1} - \omega^2 \end{vmatrix} = 0.$$

We solve the characteristic equation:

$$(-5-\omega^2)(-2-\omega^2) - 4 = 0.$$

$$10 + 7\omega^2 + \omega^4 - 4 = 0.$$

$$\omega^4 + 7\omega^2 + 6 = 0.$$

$$(\omega^2+6)(\omega^2+1) = 0.$$

$\omega^2 = -6$ or -1. Now we solve for the eigenvectors corresponding to each eigenvalue.

$$-6\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$x_1 + 2x_2 = 0$. So an eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ to correspond to $\omega^2 = -6$.

$$-1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$-4x_1 + 2x_2 = 0$. So an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, perpendicular to $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, to correspond to $\omega^2 = -1$.

Now we can look at what these eigenvalues and eigenvectors mean in the problem. It is a property of differential equations of the type with which we began that sums of solutions are solutions. Accordingly, $\omega^2 = -6$ gives $\omega = \pm i\sqrt{6}$, for both of which the corresponding eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and the corresponding solutions, recalling the presence of the exponential functions, are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ae^{i\sqrt{6}t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + Be^{-i\sqrt{6}t}\begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

where A and B are arbitrary constants that will be determined by the initial conditions of the problem. The arbitrariness of these constants is not something that should trouble you; they can in principle be complex. We need to recall that the complex exponential function is just an abbreviation for imaginary numbers in polar form, so that the equation above has the interpretation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A(\cos\sqrt{6}t + i\sin\sqrt{6}t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + B(\cos\sqrt{6}t - i\sin\sqrt{6}t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Likewise for $\omega^2 = -1$,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C(\cos t + i\sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + D(\cos t - i\sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where C and D are also arbitrary constants. Sums of solutions are solutions, and so these two solutions can be added together. In second and third steps we can also combine the arbitrary constants in a way that has a simplifying effect:

$$\begin{aligned}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= (A\cos\sqrt{6}t + Ai\sin\sqrt{6}t + B\cos\sqrt{6}t - Bi\sin\sqrt{6}t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&\quad + (C\cos t + Ci\sin t + D\cos t - Di\sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= [(A+B)\cos\sqrt{6}t + (Ai - Bi)\sin\sqrt{6}t]\begin{bmatrix} 2 \\ -1 \end{bmatrix} + [(C+D)\cos t + (Ci - Di)\sin t]\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= (a\cos\sqrt{6}t + b\sin\sqrt{6}t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (c\cos t + d\sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix},
\end{aligned}$$

where a, b, c, and d are again arbitrary constants. They are no more arbitrary for being combinations of the arbitrary constants A, B, C, and D. While a, b, c, and d are, in principle, complex constants, with real initial data like what follow they will always be real. So the use of complex numbers here is a detour to make things possible that are not so easy without them. This was, incidentally, the way complex numbers were originally invented; it was to find real roots of cubic equations not to find imaginary roots of $x^2 = -1$. No one yet wanted roots of that equation, since it obviously had none.

Solution in a specific situation depends on the initial conditions $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$ and $\begin{bmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \end{bmatrix}$. For $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (m_2 pulled down twice as far as m_1 and both released at rest), one must solve

$$\mathbf{y}(0) = (a\cos 0 + b\sin 0)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (c\cos 0 + d\sin 0)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\dot{\mathbf{y}}(0) = (-a\sqrt{6}\sin 0 + b\sqrt{6}\cos 0)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-c\sin 0 + d\cos 0)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The first is $2a + c = 1$ and $2c - a = 2$ with determinant -4 and solutions $a = 0$, $c = 1$. The second is $2\sqrt{6}b + d = 0$ and $-\sqrt{6}b + 2d = 0$ with determinant $5\sqrt{6}$ and so solutions $b = d = 0$. This solution of the original differential equation is

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ 2\cos t \end{bmatrix}.$$

Mass m_2 has amplitude twice that of m_1 and the same frequency.

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