UNIT FIVE

DETERMINANTS

5.1 INTRODUCTION

In unit one the determinant of a 2×2 matrix was introduced and used in the evaluation of a cross product. In this chapter we extend the definition of a determinant to any size square matrix. The determinant has a variety of applications. The value of the determinant of a square matrix A can be used to determine whether A is invertible or noninvertible. An explicit formula for A^{-1} exists that involves the determinant of A. Some systems of linear equations have solutions that can be expressed in terms of determinants.

5.2 DEFINITION OF THE DETERMINANT

Recall that in chapter one the determinant of the 2×2 matrix A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ was defined to be the number $a_{11}a_{22} - a_{12}a_{21}$ and that the notation det (A) or $|A|$ was used to represent the determinant of A. For any given $n \times n$ matrix $A = \lfloor a_{ij} \rfloor_{n \times n}$, the notation A_{ij} will be used to denote the $(n-1) \times (n-1)$ **submatrix** obtained from A by deleting the ith row and the jth column of A. The determinant of any size square matrix $A = \lfloor a_{ij} \rfloor_{n \times n}$ is defined recursively as follows.

Definition of the Determinant Let $A = \lfloor a_{ij} \rfloor_{n \times n}$ be an $n \times n$ matrix.

(1) If $n = 1$, that is $A = [a_{11}]$, then we define det (A) = a_{11}.

(2) If $n > 1$, we define det(A) = $\sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det(A_{1k})$

Example

If $A = [5]$, then by part (1) of the definition of the determinant, det (A) = 5.

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then by parts (2) and (1), det (A) = $(-1)^{1+1}(2)\det[5] + (-1)^{1+2}(3)\det[4]$

$= (1)(2)(5) + (-1)(3)(4) = 10 - 12 = -2$

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}$, then using parts (2) and (1), we calculate the det (A) as follows.

$$\det(A) = (-1)^{1+1}(2)\det\begin{bmatrix} 6 & 7 \\ 9 & 1 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 5 & 7 \\ 8 & 1 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

$$= (1)(2)(-57) + (-1)(3)(-51) + (1)(4)(-3) = -114 + 153 - 12 = 27.$$

Cofactor If A is a square matrix, the **ijth cofactor of A** is defined to be $(-1)^{i+j}\det(A_{ij})$. The notation C_{ij} will sometimes be used to denote the ijth cofactor of A.

Example Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then $C_{11} = (-1)^{1+1}\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = (1)(45 - 48) = -3$,

$C_{12} = (-1)^{1+2}\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = (-1)(36-42) = 6$ and $C_{23} = (-1)^{2+3}\det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = (-1)(8-14) = 6$.

In the definition of the determinant, part (2) consists of multiplying each first row entry of A by its cofactor and then summing these products. For this reason it is called a first row cofactor expansion.

Example Let $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{bmatrix}$. Use a first row cofactor expansion to evaluate det(A).

Solution det(A) =

$$(-1)^{1+1}(2)\det\begin{bmatrix} 3 & 2 & 4 \\ 3 & 4 & 1 \\ 2 & 3 & 5 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 1 & 2 & 4 \\ 5 & 4 & 1 \\ 4 & 3 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 1 & 3 & 4 \\ 5 & 3 & 1 \\ 4 & 2 & 5 \end{bmatrix} + (-1)^{1+4}(5)\det\begin{bmatrix} 1 & 3 & 2 \\ 5 & 3 & 4 \\ 4 & 2 & 3 \end{bmatrix}$$

$$=(1)(2)\left\{(-1)^{1+1}(3)\det\begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^{1+2}(2)\det\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\right\}$$

$$+(-1)(3)\left\{(-1)^{1+1}(1)\det\begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^{1+2}(2)\det\begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}\right\}$$

$$+(1)(4)\left\{(-1)^{1+1}(1)\det\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}\right\}$$

$$+(-1)(5)\left\{(-1)^{1+1}(1)\det\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix} + (-1)^{1+3}(2)\det\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}\right\}$$

$=(1)(2)\{(1)(3)(20-3) + (-1)(2)(15-2) + (1)(4)(9-8)\}$
$+(-1)(3)\{(1)(1)(20-3) + (-1)(2)(25-4) + (1)(4)(15-16)\}$
$+(1)(4)\{(1)(1)(15-2) + (-1)(3)(25-4) + (1)(4)(10-12)\}$
$+(-1)(5)\{(1)(1)(9-8) + (-1)(3)(15-16) + (1)(2)(10-12)\}$
$=(1)(2)\{51-26+4\} + (-1)(3)\{17-42-4\} + (1)(4)\{13-63-8\} + (-1)(5)\{1+3-4\}$
$=58 + 87 - 232 + 0 = -87$

Although the definition of the determinant uses a first row cofactor expansion, the determinant of A may be calculated by taking any row (or column) and multiplying the entries of that row (or column) by their cofactors and summing the products. This result is given in the next theorem whose proof is omitted.

Theorem Let A be a square $n \times n$ matrix, then

$$\det(\mathrm{A}) = \underbrace{\sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(\mathrm{A}_{ik})}_{\mathrm{i}^{\mathrm{th}}\text{ row cofactor expansion}} = \underbrace{\sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(\mathrm{A}_{kj})}_{\mathrm{j}^{\mathrm{th}}\text{ column cofactor expansion}} .$$

Example Let $\mathrm{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$. Evaluate det (A) by

(a) a second row cofactor expansion.
(b) a third column cofactor expansion.

Solution

(a) $\det(\mathrm{A}) = (-1)^{2+1}(2)\det\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} + (-1)^{2+2}(0)\det\begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix} + (-1)^{2+3}(1)\det\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

$= (-1)(2)(-4) + (1)(0)(-12) + (-1)(1)(-8) = 8 + 0 + 8 = 16.$

(b) $\det(\mathrm{A}) = (-1)^{1+3}(4)\det\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} + (-1)^{2+3}(1)\det\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + (-1)^{3+3}(0)\det\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

$= (1)(4)(2) + (-1)(1)(-8) + (1)(0)(-6) = 8 + 8 + 0 = 16.$

Theorem If A is a square matrix containing a row (or column) of zeros, then det(A) = 0.

Proof Use a cofactor expansion along the row (or column) of zeros.

Theorem If A is an $n \times n$ matrix with two identical rows (or columns), then det (A) = 0.

Proof The theorem is certainly true for $n = 2$ since $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix} = a_{11}a_{12} - a_{11}a_{12} = 0$.

If $n = 3$, use a cofactor expansion along the row different from the two identical rows. Let this row be the k$^{\text{th}}$ row. Using a cofactor expansion along this row gives
$\det(\mathrm{A}) = (-1)^{k+1}(a_{k1})\det(\mathrm{A}_{k1}) + (-1)^{k+2}(a_{k2})\det(\mathrm{A}_{k2}) + (-1)^{k+3}(a_{k3})\det(\mathrm{A}_{k3})$. But each of the submatrices A_{k1}, A_{k2} and A_{k3} has two identical rows so their determinants are 0, hence det (A) = 0 for any 3×3 matrix.
If $n > 3$ proceed as above writing det (A) as a sum of products involving submatrices with two identical rows whose determinants are 0.

Triangular and Diagonal Matrices A square matrix is said to be an **upper triangular matrix** if all the entries below the main diagonal are zero. A square matrix is said to be a **lower triangular matrix** if all the entries above the main diagonal are zero. A square matrix is said to be a **diagonal matrix** if all entries not on the main diagonal are zero. A diagonal matrix is both upper triangular and lower triangular.

Example

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ is a lower triangular matrix.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix. It is both upper and lower triangular.

Theorem If A is upper triangular, lower triangular or diagonal, $\det(A) = a_{11}a_{22}\cdots\cdots a_{nn}$.

Proof Suppose A is upper triangular. To evaluate det(A) use a cofactor expansion along the first column. Since there is only one nonzero entry in the first column the expansion gives $\det(A) = (-1)^{1+1}a_{11}\det(A_{11}) = a_{11}\det(A_{11})$. Now A_{11} is upper triangular so proceed as above to use a cofactor expansion along its first column to get $\det(A_{11}) = a_{22}\det(A_{22})$ where A_{22} is A_{11} with its first row and first column deleted. Combining the results gives $\det(A) = a_{11}a_{22}\det(A_{22})$. Continuing in this fashion, we eventually get $\det(A) = a_{11}a_{22}\cdots\cdots a_{nn}$ as required. If A is lower triangular or diagonal, the argument is similar.

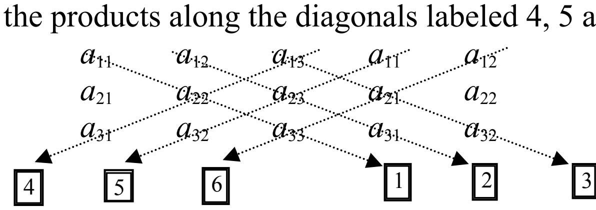
Example $\det\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} = (2)(5)(7) = 70$. $\det\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = (2)(3)(5) = 30$

Theorem $\det(I_n) = 1$ for all n.

Proof Since I_n is a diagonal matrix, $\det(I_n) = (1)(1)\cdots\cdots(1) = 1$

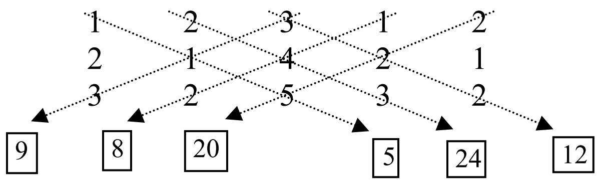
Basket-Weave Method The following method is an alternative way to evaluate the determinant of a 3×3 matrix. This method is only applicable to 3×3 matrices and is sometimes called the basket-weave method.

Construct a 3×5 array by writing down the entries of the 3×3 matrix and then repeating the first two columns. Calculate the products along the six diagonal lines shown in the diagram. The determinant is equal to the sum of products along diagonals labeled 1, 2 and 3 minus the sum of the products along the diagonals labeled 4, 5 and 6.



Example Use the basket-weave method to calculate the determinant of A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$.

Solution



$$\det(A) = (5 + 24 + 12) - (9 + 8 + 20) = (41) - (37) = 4$$

5.2 PROBLEMS

1. Use the definition of the determinant to evaluate the determinant of the given matrix.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 5 & 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & 2 & 0 & 1 \\ 5 & 0 & 3 & 2 \\ 0 & 4 & 0 & 2 \\ 3 & 1 & 0 & 2 \end{bmatrix}$ (g) $\begin{bmatrix} 5 & 4 & 3 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 2 & 4 \\ 0 & 2 & 3 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 2 & 3 & 1 & 0 \\ 4 & 1 & 0 & 3 \end{bmatrix}$

2. Use the basket-weave method to evaluate the determinants 1(a), 1(b), 1(c) and 1(d).

3. Evaluate the determinants of the following matrices by inspection.

(a) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 5 & 9 \\ 0 & 6 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 0 \\ 1 & 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

4. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 6 & 4 & 1 \end{bmatrix}$. Find the following cofactors of A.

(a) C_{11} (b) C_{12} (c) C_{13} (d) C_{21} (e) C_{22} (f) C_{32}

5. Find all values of k for which $\det\begin{bmatrix} 1\text{-}k & 3 \\ 2 & 2-k \end{bmatrix} = 0$.

5.3 ELEMENTARY ROW OPERATIONS ON DETERMINANTS

The evaluation of the determinant of an $n \times n$ matrix using the definition involves the summation of $n!$ terms, each term being a product of n factors. As n increases, this computation becomes too cumbersome and so another technique has been devised to evaluate the determinant. This technique uses the elementary row operations to reduce the matrix to a triangular form. The effect of each elementary row operation on the value of the determinant is taken into account and then the determinant of the triangular matrix is evaluated by finding the product of the entries on the main diagonal.

Theorem If A and B are square matrices and B is obtained from A by interchanging two rows (or columns) of A, then det (B) = – det (A).

Proof Let A and B be $n \times n$ matrices. If $n = 2$, then A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and B = $\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$
so det(B) = $a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21}) = -\det(A)$.
If $n = 3$, then we use a cofactor expansion for B along the row that was not interchanged. Let this be row k. Then $\det(B) = (-1)^{k+1}a_{k1}\det(B_{k1}) + (-1)a_{k2}\det(B_{k2}) + (-1)a_{k3}\det(B_{k3})$.
Each submatrix B_{kj} is the submatrix A_{kj} with its rows interchanged so $\det(B_{kj}) = -\det(A_{kj})$.
Hence $\det(B) = (-1)^{k+1}a_{k1}(-1)\det(A_{k1}) + (-1)a_{k2}(-1)\det(A_{k2}) + (-1)a_{k3}(-1)\det(A_{k3})$
$= (-1)\,[(-1)^{k+1}a_{k1}\det(A_{k1}) + (-1)a_{k2}\det(A_{k2}) + (-1)a_{k3}\det(A_{k3})]$
$= (-1)\det(A) = -\det(A)$ as required.
If $n > 3$ proceed as above using cofactor expansions along rows that were not interchanged to get the final result. The proof for interchanged columns is similar.

Theorem Let A and B be $n \times n$ matrices with B obtained from A by multiplying all the entries of some row (or column) of A by a scalar k. Then det (B) = k det (A).

Proof Suppose B is obtained from A by multiplying the entries of the j$^{\text{th}}$ row of A by k. Use a cofactor expansion for B along its j$^{\text{th}}$ row to evaluate the determinant of B and noting that these cofactors for B are equal to the corresponding cofactors for A we get
$\det(B) = (-1)^{j+1}ka_{j1}\det(B_{j1}) + (-1)^{j+2}ka_{j2}\det(B_{j2}) + \cdots\cdots + (-1)^{j+n}ka_{jn}\det(B_{jn})$
$= (-1)^{j+1}ka_{j1}\det(A_{j1}) + (-1)^{j+2}ka_{j2}\det(A_{j2}) + \cdots\cdots + (-1)^{j+n}ka_{jn}\det(A_{jn})$
$= k[(-1)^{j+1}a_{j1}\det(A_{j1}) + (-1)^{j+2}a_{j2}\det(A_{j2}) + \cdots\cdots + (-1)^{j+n}a_{jn}\det(A_{jn})]$
$= k\det(A)$.
The proof in the case where B is obtained from A by multiplying a column of A by k is similar.

5 is a common factor in row 2 of the given matrix.

Example $\det\begin{bmatrix} 7 & 3 \\ 5 & 10 \end{bmatrix} = 5\det\begin{bmatrix} 7 & 3 \\ 1 & 2 \end{bmatrix} = (5)(14-3) = 55$

Corollary det (kA) = k^n det (A).

Proof Since all n rows of A are multiplied by the scalar k to get kA, using the above theorem n times gives det (kA) $= (k)(k)\cdots\cdots\cdots(k)$ det (A) = k^n det (A).

3 is factor in row 1 and row 2

Example $\det\begin{bmatrix} 12 & 6 \\ 3 & 9 \end{bmatrix} = (3)(3)\det\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = 3^2 \det\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = (9)(12-2) = (9)(10) = 90$

Example Let A be a 3×3 matrix and let det(A) = 5. Find det(2A).

Solution $\det(2A) = 2^3 \det(A) = (8)(5) = 40$

since matrix has 2 identical rows

Example $\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 5 & 6 \end{bmatrix} = 3\det\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = (3)(0) = 0$. Notice that row 2 of the original matrix is 3 times row 1. This leads to the following corollary.

Corollary If A is a square matrix that has a row (or column) that is a scalar multiple of another row (or column), then det (A) = 0.

Proof Suppose the j$^{\text{th}}$ row of A is k times the i$^{\text{th}}$ row of A. Then det (A) = k det ($\hat{A}$) where $\hat{A}$ is the matrix A with the j$^{\text{th}}$ row multiplied by $1/k$. But $\hat{A}$ has two identical rows, (row i = row j), so det($\hat{A}$) = 0. Hence det (A) = k det($\hat{A}$) = $k \cdot 0 = 0$.

Theorem Let A be a square matrix and let B be the matrix obtained from A by adding a multiple of one row (or column) of A to another row (or column) of A.
Then det (B) = det (A).

Proof Suppose B is obtained from A by adding c times row i to row j.
Evaluate det (B) using a cofactor expansion along row j. Then

$$\det(B) = \sum_{k=1}^{n} (-1)^{j+k} b_{jk} \det(B_{jk}) = \sum_{k=1}^{n} (-1)^{j+k} (ca_{ik} + a_{jk}) \det(B_{jk})$$

$$= \sum_{k=1}^{n} (-1)^{j+k} (ca_{ik} + a_{jk}) \det(A_{jk}) \qquad \text{since } B_{jk} = A_{jk} \text{ for all k.}$$

$$= c\sum_{k=1}^{n} (-1)^{j+k} a_{ik} \det(A_{jk}) + \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det(A_{jk})$$

$= c$ det ($\hat{A}$) + det (A) where $\hat{A}$ is obtained from A by replacing the j$^{\text{th}}$ row of A by its i$^{\text{th}}$ row.

$= c \cdot 0$ + det (A) det ($\hat{A}$) = 0 because $\hat{A}$ has two identical rows

= det (A)

Example Evaluate $\det\begin{bmatrix}2&6&9\\1&2&4\\3&6&15\end{bmatrix}$

Solution

$$\det\begin{bmatrix}2&6&9\\1&2&4\\3&6&15\end{bmatrix} = \underbrace{(-1)}_{\text{rows 1 \& 2 interchanged}}\det\begin{bmatrix}1&2&4\\2&6&9\\3&6&15\end{bmatrix} = (-1)\underbrace{(3)}_{\text{row 3 has factor 3}}\det\begin{bmatrix}1&2&4\\2&6&9\\1&2&5\end{bmatrix} = \underbrace{(-1)(3)}_{\text{multiples of row 1 added to rows 2 \& 3}}\det\begin{bmatrix}1&2&4\\0&2&1\\0&0&1\end{bmatrix}$$

$$= (-1)(3)\underbrace{(1)(2)(1)}_{\text{product of entries on main diagonal}} = -6.$$

The previous example outlines an efficient technique using elementary row transformations to evaluate the determinant of a square matrix. The procedure consists of using elementary row transformations to transform the given matrix into a triangular matrix (in the above example into an upper triangular matrix), taking into account the effect of each transformation, then finally evaluating the determinant of the resulting triangular matrix by multiplying the entries along the main diagonal.

5.3 PROBLEMS

1. Use elementary row operations to evaluate the determinants of the following matrices.

(a) $\begin{bmatrix}2&3&4\\1&4&3\\3&6&9\end{bmatrix}$ (b) $\begin{bmatrix}3&1&5\\2&6&8\\1&3&2\end{bmatrix}$ (c) $\begin{bmatrix}4&8&4\\2&6&8\\3&9&6\end{bmatrix}$ (d) $\begin{bmatrix}1&2&3\\6&3&9\\4&2&8\end{bmatrix}$

(e) $\begin{bmatrix}2&3&5&4\\1&3&0&2\\2&4&6&4\\3&6&3&0\end{bmatrix}$ (f) $\begin{bmatrix}1&2&3&4\\1&4&4&7\\2&4&1&5\\3&6&9&2\end{bmatrix}$ (g) $\begin{bmatrix}3&1&2&4\\6&5&3&7\\9&3&2&1\\6&2&4&5\end{bmatrix}$

5.4 PROPERTIES OF DETERMINANTS

Let A be a square matrix. Let $\hat{A}$ be the matrix resulting from performing one or more elementary row operation on A. Since the effect of performing an elementary row operation on the value of the determinant is either to reverse the sign or multiply the value of the determinant by a nonzero number, and since the elementary row operations are invertible operations; therefore $\det(\hat{A}) \neq 0$ if and only if $\det(A) \neq 0$ and similarly $\det(\hat{A}) = 0$ if and only if $\det(A) = 0$.

Let a sequence of elementary row operations be performed on the $n \times n$ matrix A so as to reduce A to its reduced row-echelon form R. Now A is invertible if and only if R = I. But det (R) = det (I) = 1 $\neq 0$ if and only if det (A) $\neq 0$. We therefore conclude that A is invertible if and only if $\det(A) \neq 0$ and state this result in the form of a theorem.

Theorem The square matrix A is invertible if and only if det (A) $\neq 0$.

Example $\det\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 6 - 8 = -2 \neq 0$ so $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ is an invertible matrix.

$\det\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$ so $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not invertible.

A direct consequence of the above theorem is the following result.

Theorem Let A be a square matrix. Then the linear system A**x** = **b** has a unique solution for every **b** if and only if det (A) $\neq 0$.

Proof Suppose det (A) $\neq 0$, then A is invertible. Then $A^{-1}\mathbf{b}$ is a solution to A**x** = **b** since $A(A^{-1}\mathbf{b}) = \mathbf{b}$. To show that this is the only solution to A**x** = **b,** suppose that $\mathbf{x_0}$ is also a solution to A**x** = **b.** Then $A\mathbf{x_0} = \mathbf{b}$ so $A^{-1}(A\,\mathbf{x_0}) = A^{-1}\mathbf{b}$ and hence $\mathbf{x_0} = A^{-1}\mathbf{b}$. This shows that $A^{-1}\mathbf{b}$ is the unique solution to A**x** = **b**.
On the other hand, if A**x** = **b** has a unique solution, then when solving this system by matrix methods the coefficient matrix is reduced to the identity matrix I and so A is invertible and hence $\det(A) \neq 0$.

Theorem Let A be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix.
Then det(EA) = det(E) det(A).

Proof The proof consists in showing that the result is true for each one of the three types of elementary matrices.
Let E be the elementary matrix obtained from I by interchanging two rows of I. Then EA is the matrix resulting from interchanging the corresponding two rows of A.
Then det(EA) = – det(A) = (–1) det(A) = det(E) det(A) since det(E) = –1.
Let E be the elementary matrix obtained from I by multiplying the entries of some row of I by a nonzero scalar *k.* Then EA is the matrix resulting from multiplying the entries of a row of A by *k*. Then det(EA) = *k* det(A) = det(E) det(A) since det(E) = *k.*
Let E be the elementary matrix obtained from I by adding a multiple of one row of I to another row of I. Then EA is the result of adding a multiple of a row of A to another row of A. Then det(EA) = det(A) = (1) det(A) = det(E) det(A) since det(E) = 1.

The result in the above theorem can be generalized to any two $n \times n$ matrices. The proof is omitted but stated in the following theorem.

Theorem If A and B are square matrices of the same size, then det(AB) = det(A) det(B).

Example Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, and let $B = \begin{bmatrix} 6 & 7 \\ 5 & 8 \end{bmatrix}$, then $AB = \begin{bmatrix} 39 & 52 \\ 16 & 23 \end{bmatrix}$.

det(A) = 8 – 3 = 5 det(B) = 48 – 35 = 13 and det (AB) = 897 – 832 = 65

det(A) det(B) = (5)(13) = 65 = det(AB).

Theorem If A is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$

Proof $A^{-1}A = I \Rightarrow \det(A^{-1}A) = \det(I) \Rightarrow \det(A^{-1})\det(A) = 1 \Rightarrow \det(A^{-1}) = 1/\det(A)$.

Theorem If A is a square matrix, then $\det(A^T) = \det(A)$.

Proof A cofactor expansion along the first row of A^T gives the same terms as a cofactor expansion along the first column of A.

Example Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. det(A) = (2)(3) – (1)(4) = 6 – 4 = 2.
$\det(A^T)$ = (2)(3) – (4)(1) = 6 – 4 = 2. So det (A) = $\det(A^T)$ = 2.

Theorem Let A and B be square matrices with AB = I. Then BA = I.

Proof We first show that there exists a matrix C such that CA = I and then show that in fact C = B.
Since AB = I and since $\det(I) \neq 0$, therefore $\det(A) \neq 0$. But $\det(A^T) = \det(A)$, so $\det(A^T) \neq 0$ and hence A^T is invertible. Let D denote the inverse of A^T; so $D = (A^T)^{-1}$.
Then $A^TD = I \Rightarrow (A^TD)^T = I^T \Rightarrow D^T(A^T)^T = I^T \Rightarrow D^TA = I$ so $D^T = C$.
We now show C = B as follows. C = CI = C(AB) = (CA)B = IB = B.

When computing the inverse of a matrix A one should verify the correctness of the computation by demonstrating that both the products AA^{-1} and $A^{-1}A$ equal I. The preceding theorem proves that in fact that it is sufficient to show that only one of these two products needs to be shown equal to I.

5.4 PROBLEMS

1. Determine whether the matrix is invertible or not by calculating its determinant.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix}$

2. Use determinants to show that the following systems of linear equations have unique solutions.

(a) $\begin{aligned} x+2y&=3 \\ 3x+4y&=1 \end{aligned}$ (b) $\begin{aligned} 3x-4y&=7 \\ 2x+5y&=9 \end{aligned}$ (c) $\begin{aligned} x+2y+3z&=5 \\ 2x+y+4z&=3 \\ 3x+4y+7z&=6 \end{aligned}$ (d) $\begin{aligned} 2x+3y+z&=2 \\ x+2y+3z&=4 \\ 3x+5y+2z&=5 \end{aligned}$

3. Let A and B be 2×2 matrices with det(A) = 3 and det(B) = 4. Find the following.

(a) det(AB) (b) det(A^2) (c) det(AB^{-1}) (d) $\det(AB)^{-1}$ (e) det(A^TB)

4. Let A and B be 3×3 matrices with det(A) = 2 and det(B) = 5. Find the following.

(a) det(AB) (b) det(A^2) (c) det(AB^{-1}) (d) $\det(AB)^{-1}$ (e) det(A^TB)

5.5 THE ADJOINT MATRIX

Recall the kth row cofactor expansion of an $n\times n$ matrix A for $n\geq 2$ was defined to be $\det(A)=a_{k1}(-1)^{k+1}\det(A_{k1})+a_{k2}(-1)^{k+2}\det(A_{k2})+\cdots\cdots\cdots+a_{kn}(-1)^{k+n}\det(A_{kn})$ where the quantity $(-1)^{k+j}\det(A_{kj})$ is called the kjth cofactor of A. To simplify our notation we will denote this quantity by the symbol C_{kj}. Thus the kth row cofactor expansion for det(A) can be written more simply as det(A) = $a_{k1}C_{k1}+a_{k2}C_{k2}+\cdots\cdots+a_{kn}C_{kn}$. Suppose that in this expression we replace the kth row entries $a_{k1},a_{k2}\cdots\cdots,a_{kn}$ by the jth row entries $a_{j1},a_{j2}\cdots\cdots,a_{jn}$ to get $a_{j1}C_{k1}+a_{j2}C_{k2}+\cdots\cdots+a_{jn}C_{kn}$. Such an expression would arise if the entries of the kth row of A were replaced by the entries of the jth row of A and a cofactor expansion along this new kth row were done. But the value of this determinant would be 0 since the matrix has two identical rows (rows k and j are same). We have thus established the following theorem.

Theorem If A is a square matrix, then $\displaystyle\sum_{k=1}^{n} a_{ik}C_{jk}=\begin{cases}\det(A) & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$

In a similar fashion, we can deduce the following result for a column cofactor expansion.

If a cofactor expansion along a column is used then $\sum_{k=1}^{n} a_{ki}\mathrm{C}_{kj} = \begin{cases} \det(\mathrm{A}) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Cofactor Matrix Let A be a square matrix. The **cofactor matrix** of A, denoted **cof(A)** is the matrix obtained from A by replacing every entry of A by its cofactor.

Example

$$\text{If A} = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 8 & 7 \\ 0 & 3 & 4 \end{bmatrix}, \text{ then cof(A)} = \begin{bmatrix} \begin{vmatrix} 8 & 7 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 5 & 7 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 5 & 8 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 6 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 6 \\ 8 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 6 \\ 5 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 8 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 11 & -20 & 15 \\ 10 & 4 & -3 \\ -34 & 23 & -2 \end{bmatrix}.$$

Adjoint Matrix If A is a square matrix, the **adjoint of A**, denoted **adj(A)** is the transpose of the cofactor matrix; that is adj(A) = $[\mathrm{cof(A)}]^{\mathrm{T}}$.

Example Continuing with the previous example,

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 8 & 7 \\ 0 & 3 & 4 \end{bmatrix}, \quad \mathrm{cof(A)} = \begin{bmatrix} 11 & -20 & 15 \\ 10 & 4 & -3 \\ -34 & 23 & -2 \end{bmatrix} \quad \text{and so } \mathrm{adj(A)} = [\mathrm{cof(A)}]^{\mathrm{T}} = \begin{bmatrix} 11 & 10 & -34 \\ -20 & 4 & 23 \\ 15 & -3 & -2 \end{bmatrix}$$

Consider now the product $\mathrm{A} \cdot \mathrm{adj(A)} = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 8 & 7 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 11 & 10 & -34 \\ -20 & 4 & 23 \\ 15 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 61 & 0 & 0 \\ 0 & 61 & 0 \\ 0 & 0 & 61 \end{bmatrix}$

But

$$\det(\mathrm{A}) = (1)\det\begin{bmatrix} 8 & 7 \\ 3 & 4 \end{bmatrix} - (2)\det\begin{bmatrix} 5 & 7 \\ 0 & 4 \end{bmatrix} + (6)\det\begin{bmatrix} 5 & 8 \\ 0 & 3 \end{bmatrix} = (1)(11) - (2)(20) + (6)(15) = 61.$$

We see that the product $\mathrm{A} \cdot \mathrm{adj(A)}$ is a diagonal matrix with the diagonal entries = det(A).

This example suggests the following theorem.

Theorem If A is a square matrix, then $A \cdot \text{adj}(A) = \det(A) \cdot I$

Proof $A \cdot \text{adj}(A) =$

$$\begin{bmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & \bullet & \bullet & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \bullet & \bullet & C_{n1} \\ C_{12} & C_{22} & \bullet & \bullet & C_{n2} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ C_{1n} & C_{2n} & \bullet & \bullet & C_{nn} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_{1k}C_{1k} & \sum_{k=1}^{n} a_{1k}C_{2k} & \bullet & \bullet & \sum_{k=1}^{n} a_{1k}C_{nk} \\ \sum_{k=1}^{n} a_{2k}C_{1k} & \sum_{k=1}^{n} a_{2k}C_{2k} & \bullet & \bullet & \sum_{k=1}^{n} a_{2k}C_{nk} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \sum_{k=1}^{n} a_{nk}C_{1k} & \sum_{k=1}^{n} a_{nk}C_{2k} & \bullet & \bullet & \sum_{k=1}^{n} a_{nk}C_{nk} \end{bmatrix} =$$

$$\begin{bmatrix} \det(A) & 0 & 0 & 0 & 0 \\ 0 & \det(A) & 0 & 0 & 0 \\ 0 & 0 & \det(A) & 0 & 0 \\ 0 & 0 & 0 & \det(A) & 0 \\ 0 & 0 & 0 & 0 & \det(A) \end{bmatrix} = \det(A) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \det(A) \cdot I$$

Theorem If $\det(A) \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{\det(A)} \cdot \text{adj}(A)$.

Proof From the previous theorem we have $A \cdot \text{adj}(A) = \det(A) \cdot I$. Since $\det(A) \neq 0$ we can divide by det(A) to get $A \cdot \dfrac{1}{\det(A)} \cdot \text{adj}(A) = I$ so $A^{-1} = \dfrac{1}{\det(A)} \cdot \text{adj}(A)$.

Example Use the preceding theorem to find A^{-1} for the matrix $A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 8 & 7 \\ 0 & 3 & 4 \end{bmatrix}$.

Solution This is the same matrix used in the previous example where we found

$\text{adj}(A) = \begin{bmatrix} 11 & 10 & -34 \\ -20 & 4 & 23 \\ 15 & -3 & -2 \end{bmatrix}$ and det(A) = 61. Using $A^{-1} = \dfrac{1}{\det(A)} \text{adj}(A)$ we get

$$A^{-1} = \frac{1}{61} \begin{bmatrix} 11 & 10 & -34 \\ -20 & 4 & 23 \\ 15 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 11/61 & 10/61 & -34/61 \\ -20/61 & 4/61 & 23/61 \\ 15/61 & -3/61 & -2/61 \end{bmatrix}.$$

5.5 PROBLEMS

1. For each of the following matrices find its adjoint, then use the adjoint (and the value of the determinant) to find the inverse of the matrix.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

2. Let A be an $n \times n$ matrix with $\det(\mathrm{A}) \neq 0$. Show that det[adj(A)] = $[\det(\mathrm{A})]^{n-1}$.

3. Let A be a 3×3 matrix with det(A) = 5. Find det[adj(A)].

4. Let A be a 4×4 matrix with det(A) = 3. Find det[adj(A)].

5. Let A be a 5×5 matrix with det(A) = 2. Find det[adj(A)].

5.6 CRAMER'S RULE

Cramer's rule provides a formula for solving a system of n linear equations in n variables when the system has a unique solution.

Theorem (Cramer's Rule) Let A**x** = **b** be a system of n linear equations in n variables with det(A) $\neq 0$. Let $\mathrm{A_k}$ be the matrix obtained from A by replacing the $\mathrm{k^{th}}$ column of A by the column vector **b**. Then the system has the unique solution

$$x_{\mathrm{k}} = \frac{\det(\mathrm{A_k})}{\det(\mathrm{A})}, \ \mathrm{k} = 1, 2, \cdots\cdots, n .$$

Proof Since det(A) $\neq 0$, A is invertible and the system A**x** = **b** has the unique solution

$\mathbf{x} = \mathrm{A}^{-1}\mathbf{b}$. Therefore $\mathbf{x} = \dfrac{1}{\det(\mathrm{A})}\mathrm{adj}(\mathrm{A}) \cdot \mathbf{b} = \dfrac{1}{\det(\mathrm{A})}\begin{bmatrix} \mathrm{C}_{11} & \mathrm{C}_{21} & \bullet & \bullet & \mathrm{C}_{\mathrm{n}1} \\ \mathrm{C}_{12} & \mathrm{C}_{22} & \bullet & \bullet & \mathrm{C}_{\mathrm{n}2} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \mathrm{C}_{1\mathrm{n}} & \mathrm{C}_{2\mathrm{n}} & \bullet & \bullet & \mathrm{C}_{\mathrm{nn}} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \bullet \\ \bullet \\ b_n \end{bmatrix}$

$$= \frac{1}{\det(A)}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots\cdots\cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots\cdots\cdots + b_nC_{n2} \\ \bullet \\ \bullet \\ b_1C_{1n} + b_2C_{2n} + \cdots\cdots\cdots + b_nC_{nn} \end{bmatrix}.$$

Thus $x_k = \dfrac{b_1C_{1k} + b_2C_{2k} + \cdots\cdots\cdots + b_nC_{nk}}{\det(A)}$ for $k = 1, 2, \cdots\cdots\cdots, n.$

Now the numerator of x_k consists of the kth column cofactors of A multiplied by the corresponding entries of **b**. We get the same result if we use a kth column cofactor expansion of A_k so $x_k = \dfrac{\det(A_k)}{\det(A)}$ for $k = 1, 2\cdots\cdots\cdots, n.$

Example Use Cramer's rule to solve the following system of linear equations.

$$x + 2y + z = 9$$
$$2x + y + z = 7$$
$$x + y + 3z = 10$$

Solution

$$x = \frac{\det\begin{bmatrix} 9 & 2 & 1 \\ 7 & 1 & 1 \\ 10 & 1 & 3 \end{bmatrix}}{\det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}} = \frac{-7}{-7} = 1 \quad y = \frac{\det\begin{bmatrix} 1 & 9 & 1 \\ 2 & 7 & 1 \\ 1 & 10 & 3 \end{bmatrix}}{\det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}} = \frac{-21}{-7} = 3 \quad z = \frac{\det\begin{bmatrix} 1 & 2 & 9 \\ 2 & 1 & 7 \\ 1 & 1 & 10 \end{bmatrix}}{\det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}} = \frac{-14}{-7} = 2$$

4.6 PROBLEMS

1. Use Cramer's rule to solve the following systems of linear equations.

(a) $2x + 3y = 8$
$x - 2y = -3$

(b) $4x + y = 13$
$x + 3y = 6$

(c) $x + 2y = 1$
$2x + y = 5$

(d) $x + y + z = 6$
$x - y + z = 0$
$x + y - z = 2$

(e) $2x + y - z = 0$
$x + 2y - z = 1$
$x + y = 2$

(f) $x + y + z = 3$
$2x + y - 3z = 2$
$3x + 4y + 2z = 7$

5.7 APPLICATIONS USING THE DETERMINANT

In this section two applications using the determinant will be considered. The first is a convenient method for evaluating a cross product. The second application is finding the volume of a parallelepiped.

Cross Product Recall if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\\ &= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

A convenient method for arriving at the cross product $\mathbf{u}\times\mathbf{v}$ is to use the matrix form $\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}$ and do a first row cofactor expansion as you would do to evaluate a determinant.

Example Let $\mathbf{u} = (3, 2, 1)$ and let $\mathbf{v} = (4, 5, 6)$. Find $\mathbf{u}\times\mathbf{v}$.

Solution $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 1\\ 4 & 5 & 6\end{vmatrix} = \begin{vmatrix}2 & 1\\ 5 & 6\end{vmatrix}\mathbf{i} - \begin{vmatrix}3 & 1\\ 4 & 6\end{vmatrix}\mathbf{j} + \begin{vmatrix}3 & 2\\ 4 & 5\end{vmatrix}\mathbf{k} = 7\mathbf{i} - 14\mathbf{j} + 7\mathbf{k} = (7, -14, 7)$

Volume of a Parallelepiped Recall that if a parallelepiped has the vectors **u, v** and **w** as edges, then the volume of the parallelepiped is $|\mathbf{w}\bullet(\mathbf{u}\times\mathbf{v})|$. A straight forward calculation shows that $\mathbf{w}\bullet(\mathbf{u}\times\mathbf{v}) = \det\begin{bmatrix}w_1 & w_2 & w_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix} = \det\begin{bmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{bmatrix}$

Hence the volume of the parallelepiped can be found by taking the absolute value of either of the above determinants.

Example Find the volume of the parallelepiped having the vectors (1, 0, 1), (2, 3, 4) and (3, 5, 5) as edges.

Solution

$$\text{Volume} = \left|\det\begin{bmatrix}1 & 0 & 1\\ 2 & 3 & 4\\ 3 & 5 & 5\end{bmatrix}\right| = |-4| = 4.$$

5.7 PROBLEMS

1. Calculate the following cross products.

(a) $(2,3,1)\times(3,1,4)$ (b) $(1,0,2)\times(3,5,4)$ (c) $(5,3,4)\times(2,0,1)$ (d) $(3,0,1)\times(2,5,4)$

2. Find the volume of the parallelepiped having the following vectors as edges.

(a) (2, 3, 1), (3, 4, 6), (5, 7, 9) (b) (1, 0, 2), (3, 5, 7), (5, 6, 8)

(c) (2, 0, 1), (5, 4, 6), (7, 8, 8) (d) (4, 6, 5), (2, 1, 3), (7, 8, 7)