UNIVERSITY OF MANITOBA

DATE: October 20, 2008 MIDTERM

TITLE PAGE

COURSE: MATH 1210 TIME: 60 minutes

EXAMINATION: Classical and Linear Algebra EXAMINER: M. Davidson

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FAMILY NAME: (Print in ink) ____________________

GIVEN NAME(S): (Print in ink) ____________________

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 60 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 5 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 40 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	11	
2	8	
3	6	
4	7	
5	8	
Total:	40	

[9] 1. (a) Prove the following, using induction, for all $n \geq 1$:

$$2+5+8+\cdots+(3n-1)=\frac{n(3n+1)}{2}$$

Solution:
Let P_n be the statement $2+5+8+\cdots+(3n-1)=\frac{n(3n+1)}{2}$.

If $n=1$:
$2+5+8+\cdots+(3n-1)=2$
and $\frac{n(3n+1)}{2}=\frac{4}{2}=2$
So P_1 is true.
Assume that P_k is true.
Then $2+5+8+\cdots+(3k-1)=\dfrac{k(3k+1)}{2}$.
Now

$$\begin{aligned}2+5+8+\cdots+(3(k+1)-1) &= 2+5+8+\cdots+(3k-1)+(3k+2)\\ &= \frac{k(3k+1)}{2}+(3k+2)\\ &= \frac{3k^2+k+6k+4}{2}\\ &= \frac{3k^2+7k+4}{2}\\ &= \frac{(k+1)(3k+4)}{2}\\ &= \frac{(k+1)(3(k+1)+1)}{2}\end{aligned}$$

Hence if P_k is true then P_{k+1} is also true.
Since P_1 is true and P_k implies P_{k+1}, by PMI, P_n is true for all $n \geq 1$.

[2] (b) Write $2+5+8+\cdots+(3n-1)$ in sigma notation.

Solution:

$$\sum_{i=1}^{n}(3i-1)$$

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[8] 2. Find all complex numbers z such that $z^3 = -4 + 4\sqrt{3}i$. Express your answer(s) in exponential form.

Solution:

$|-4+4\sqrt{3}i| = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{16+48} = \sqrt{64} = 8$

$arg(-4+4\sqrt{3}i) = \frac{2\pi}{3}$

The exponential form is $-4+4\sqrt{3}i = 8e^{i(\frac{2\pi}{3})}$

Let $z = re^{i\theta}$, so $z^3 = r^3 e^{i(3\theta)}$.

So the equation $z^3 = -4+4\sqrt{3}i$ becomes $r^3 e^{i(3\theta)} = 8e^{i(\frac{2\pi}{3})}$.

From this, we know that $r^3 = 8$ and $3\theta = \frac{2\pi}{3} + 2n\pi$.

The modulus of all three roots is $r = 2$.

The arguments of the roots are

$$\begin{aligned}\theta &= (\frac{1}{3})(\frac{2\pi}{3} + 2n\pi)\\ &= \frac{2\pi}{9} + \frac{2n\pi}{3}\\ &= \frac{(2+6n)\pi}{9}\end{aligned}$$

The arguments of the three roots can be found by finding angles for the values of $n = 0, 1, 2$.

So $\theta_0 = \dfrac{2\pi}{9}$,

$\theta_1 = \dfrac{8\pi}{9}$

and $\theta_2 = \dfrac{14\pi}{9} \left(\equiv \dfrac{-4\pi}{9}\right)$

Hence the three solutions are

$$2e^{(\frac{2\pi}{9})i},\ 2e^{(\frac{8\pi}{9})i},\ \text{and}\ 2e^{(\frac{14\pi}{9})i}.$$

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[6] 3. Consider the polynomial $P(x) = 3\,x^6 + 12\,x^5 - 4\,x^3 + 17x^2 + 5$.
(DO NOT ATTEMPT TO FACTOR THIS POLYNOMIAL)

(a) Apply Descartes rules of signs to $P(x)$. Be specific about what information it gives.

Solution:
Since there are 2 sign changes in $P(x)$, the number of positive real roots of $P(x)$ is 2 or 0.
$P(-x) = 3\,x^6 - 12\,x^5 + 4\,x^3 + 17x^2 + 5$
Since $P(-x)$ has 2 sign changes, the number of negative real roots of $P(x)$ is 2 or 0.

(b) Apply the bound theorem to $P(x)$. Be specific about what information it gives.

Solution:
If $P(\alpha) = 0$ then using the bounds theorem $|\alpha| < \frac{M}{a_4} + 1$ where $M = \max\{12, 4, 17, 5\}$.
So $|x| < \frac{17}{3} + 1 = \frac{20}{3} = 6\frac{2}{3}$
The modulus of a root of $P(x)$ must be smaller than $6\frac{2}{3}$.

(c) What are the possible rational roots of $P(x)$? Include any information from part a and/or part b.

Solution: Since 3 has divisors $\{\pm 1, \pm 3\}$ and 5 has divisors $\{\pm 1, \pm 5\}$, from the rational root theorem we know that the possible rational roots are limited to the set
$\{\pm 1, \pm 5, \pm\frac{1}{3}, \pm\frac{5}{3}\}$
None of the possibilities are beyond the bound mentioned in part b, and we know from part a that both positive and negative roots are possible. Hence the possible rational roots are
$\{\pm 1, \pm 5, \pm\frac{1}{3}, \pm\frac{5}{3}\}$

[7] 4. Given that $1 - i$ is a root of $P(x) = 3x^4 - 8x^3 + 12x^2 - 8x + 4$, express $P(x)$ as a product of linear terms.

Solution: We know that since $1 - i$ is a root, then so $1 + i$ is also a root. Hence $(x - (1 - i))$ and $(x - (1 + i))$ are both factors.

So $(x - (1 - i))(x - (1 + i)) = (x^2 - 2x + 2)$ is also a factor.

$$\begin{array}{r|l} & 3x^2 - 2x + 2 \\ \hline x^2 - 2x + 2 & 3x^4 - 8x^3 + 12x^2 - 8x + 4 \\ & \underline{3x^4 - 6x^3 + 6x^2} \\ & -2x^3 + 6x^2 - 8x \\ & \underline{-2x^3 + 4x^2 - 4x} \\ & x^2 - 4x + 4 \\ & \underline{x^2 - 4x + 4} \\ & 0 \end{array}$$

To find the roots of $3x^2 - 2x + 2$ we use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(2)}}{2(3)} \\ &= \frac{2 \pm \sqrt{4 - 24}}{6} \\ &= \frac{2 \pm \sqrt{-20}}{6} \\ &= \frac{2 \pm 2\sqrt{5}i}{6} \\ &= \frac{1 \pm \sqrt{5}i}{3} \end{aligned}$$

Hence $P(x) = 3(x - (1 - i))(x - (1 + i))(x - (\frac{1+\sqrt{5}i}{3})(x - (\frac{1-\sqrt{5}i}{3}))$.

[8] 5. Given $\overrightarrow{u} = [3, 4, -1]$ and $\overrightarrow{v} = [-2, 3, 7]$ and θ is the angle between them.

(a) Find the value of $\cos\theta$. (do not simplify)

Solution:
Using the formula $\overrightarrow{u} \cdot \overrightarrow{v} = ||\overrightarrow{u}||\,||\overrightarrow{v}|| \cos\theta$;

$$\overrightarrow{u} \cdot \overrightarrow{v} = (3)(-2) + (4)(3) + (-1)(7) = -6 + 12 - 7 = -1$$

$$||\overrightarrow{u}|| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$||\overrightarrow{v}|| = \sqrt{(-2)^2 + 3^2 + 7^2} = \sqrt{4 + 9 + 48} = \sqrt{62}$$

Giving $-1 = \sqrt{26}\sqrt{62}\cos\theta$.
So $\cos\theta = \dfrac{-1}{\sqrt{26}\sqrt{62}}$.

(b) Find a non-zero vector that is orthogonal to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

Solution:
A vector that is orthogonal to both $\overrightarrow{u}$ and $\overrightarrow{v}$ is $\overrightarrow{u} \times \overrightarrow{v}$.
The formula is $\overrightarrow{u} \times \overrightarrow{v} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]$.
So

$$\begin{aligned}\overrightarrow{u} \times \overrightarrow{v} &= [(4)(7) - (3)(-1), (-1)(-2) - (3)(7), (3)(3) - (-2)(4)] \\ &= [28 + 3, 2 - 21, 9 + 8] \\ &= [31, -19, 17]\end{aligned}$$

So a nonzero vector that is othogonal to both $\overrightarrow{u}$ and $\overrightarrow{v}$ is $[31, -19, 17]$.