Mathematics 1210 Assignment 2 Fall 2010

1. Simplify $\dfrac{169}{5+12i} + \left(\overline{(1-2i)^3+4}\right)^2$ and express in Cartesian form.

Solution:

$$\frac{169(5-12i)}{(5+12i)(5-12i)} + \left(\overline{1-2\cdot 3i+3\cdot 4i^2-8i^3+4}\right)^2 = \frac{169(5-12i)}{169} + \left(\overline{-7+2i}\right)^2$$
$$= (5-12i) + (-7-2i)^2 = (5-12i) + (45+28i) = 50+16i$$

2. Express in the forms required, with all arguments in your answers reduced to numbers in the interval $(-\pi, \pi]$.

   (a) $-6+\sqrt{108}i$ in polar and exponential form;

   (b) $\sqrt{18}\left(\cos\dfrac{19\pi}{4}+i\sin\dfrac{19\pi}{4}\right)$ in Cartesian and exponential form;

   (c) $10e^{-\frac{5\pi}{6}i}$ in Cartesian and polar form.

Solution: (a)

$$\begin{aligned}
-6+\sqrt{108}i &= 12\left(\frac{-6}{12}+\frac{\sqrt{108}}{12}i\right) = 12\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)\\
&= 12\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) \quad \text{(polar)}\\
&= 12e^{\frac{2\pi}{3}i} \quad \text{(exponential)}
\end{aligned}$$

(b)

$$\begin{aligned}
\sqrt{18}\left(\cos\frac{19\pi}{4}+i\sin\frac{19\pi}{4}\right) &= \sqrt{18}e^{\frac{19\pi}{4}i}\\
&= \sqrt{18}e^{\frac{3\pi}{4}i} \quad \text{(exponential -- or } 3\sqrt{2}e^{\frac{3\pi}{4}i}\text{)}\\
&= 3\sqrt{2}\left(\frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 3(-1+i) = -3+3i \quad \text{(Cartesian)}
\end{aligned}$$

(c)

$$\begin{aligned}
10e^{-\frac{5\pi}{6}i} &= 10\left(\cos\frac{-5\pi}{6}+i\sin\frac{-5\pi}{6}\right) \quad \text{(polar)}\\
&= 10\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right) = -5\sqrt{3}-5i \quad \text{(Cartesian)}
\end{aligned}$$

3. $\cos n\theta$, $n \in \mathbb{Z}$, can always be expressed in terms of $\sin\theta$ and $\cos\theta$. For example, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$. Use De Moivre's Theorem to obtain an expression of this type for $\cos 7\theta$. (HINT: look at the real part of a complex number.)

Solution: By de Moivre's Theorem we have

$$\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta.$$

Therefore,

$$\begin{aligned}
\cos 7\theta &= \mathrm{Re}(\cos\theta + i\sin\theta)^7 \\
&= \mathrm{Re}(\cos^7\theta + 7\cos^6\theta i\sin\theta + 21\cos^5\theta i^2\sin^2\theta + 35\cos^4\theta i^3\sin^3\theta \\
&\quad + 35\cos^3\theta i^4\sin^4\theta + 21\cos^2\theta i^5\sin^5\theta + 7\cos\theta i^6\sin^6\theta + i^7\sin^7\theta) \\
&= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^{3\theta}\sin^4\theta - 7\cos\theta\sin^2\theta
\end{aligned}$$

4. Find all of the complex 6th roots of -64. Express your answers in Cartesian form.

Solution:

$$\begin{aligned}(-64)^{\frac{1}{6}} &= \left(64e^{\pi i}\right)^{\frac{1}{6}} = \left(2^6 e^{(2k+1)\pi i}\right)^{\frac{1}{6}} \\ &= 2e^{\frac{(2k+1)\pi}{6}}, \qquad k = 0, 1, 2, 3, 4, 5\end{aligned}$$

Which give, respectively, $\sqrt{3}+i$, $2i$, $-\sqrt{3}+i$, $-\sqrt{3}-i$, $-2i$, $\sqrt{3}-i$.

5. Solve the equation $x^4 - 8x^2 + 36 = 0$ over the complex numbers.

Solution: Let $z = x^2$. First we solve $z^2 - 8z + 36 = 0$:

$$z = \frac{8 \pm \sqrt{64 - 144}}{2} = 4 \pm i\sqrt{20} = 6\left(\frac{2}{3} \pm i\frac{\sqrt{5}}{3}\right) = 6(\cos\theta \pm i\sin\theta) = 6e^{\pm i\theta}.$$

From the root $z = 6e^{i\theta}$, we obtain solutions

$$x = z^{\frac{1}{2}} = \left(6e^{(\theta + 2k\pi)i}\right)^{\frac{1}{2}} = \sqrt{6}e^{\left(\frac{\theta}{2} + k\pi\right)i}, \qquad k = 0, 1$$

That is, $x = \pm\sqrt{6}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. By the half-angle identities,

$$\cos\frac{\theta}{2} = \sqrt{\frac{1 + \cos\theta}{2}} = \sqrt{\frac{1 + \frac{2}{3}}{2}} = \sqrt{\frac{5}{6}}$$

and

$$\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}} = \sqrt{\frac{1 - \frac{2}{3}}{2}} = \sqrt{\frac{1}{6}},$$

so that $x = \sqrt{6}\left(\sqrt{\frac{5}{6}} + i\sqrt{\frac{1}{6}}\right) = \pm(\sqrt{5} + i)$.

For $z = 6e^{-i\theta} = 6(\cos\theta - i\sin\theta)$, the difference is that in the half angle theorem we must use $\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}}$ because θ is in the fourth quadrant. So we obtain $x = \pm\sqrt{6}\left(\sqrt{\frac{5}{6}} - i\sqrt{\frac{1}{6}}\right) = \pm(\sqrt{5} - i)$.

So the roots of the equation are $\sqrt{5} + i$, $\sqrt{5} - i$, $-\sqrt{5} + i$ and $-\sqrt{5} - i$.

6. (a) Use long division to find the quotient and remainder when $x^5 - 3x^4 + 2x^2 - x + 7$ is divided by $x - 3$. Express the result as an equation in the form (polynomial) = (polynomial)·(quotient)+(remainder).

(b) Use the Remainder Theorem to find the remainder when $f(x) = (1+i)x^4 + 3ix^3 + (1-i)x + 2$ is divided by $ix - 3$. (Do not perform long division!)

(c) For which value of d is the polynomial $2x - 3$ a factor of the polynomial $g(x) = x^3 - 5x^2 + 2x - d$?

(d) You are given that $(x-2)$ and $(x+1)$ are factors of the polynomial $f(x) = x^4 - 8x^3 + hx^2 + kx + 6$. What are the values of h and k?

Solution: (a) $x^5 - 3x^4 + 2x^2 - x + 7 = (x-3)(x^4 + 2x + 5) + 22$

(b) $ix - 3 = i(x - \frac{3}{i}) = i(x - (-3i))$, so the remainder is $f(-3i) = (1+i)(-3i)^4 + 3i(-3i)^3 + (1-i)(-3i) + 2 = -1 + 78i$

(c) $2x - 3 = 2\left(x - \frac{3}{2}\right)$, which is a factor if and only if $g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 - 5\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) - d = \frac{-39}{8} - d = 0$, that is if and only if $d = -\frac{39}{8}$.

(d) Using the Remainder Theorem we obtain

$$\begin{aligned} f(2) &= 2^4 - 8 \cdot 2^3 + h \cdot 2^2 + k \cdot 2 + 6 = -42 + 4h + 2k = 0, \\ f(-1) &= 1 + 8 + h - k + 6 = 15 + h - k = 0. \end{aligned}$$

So $2h + k = 21$ and $h - k = -15$. Adding, we obtain $3h = 6$, so $h = 2$, and $k = 21 - 4 = 17$.

7. You are given that $2+i$ is a zero of the polynomial $p(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$. Write $p(x)$ as a product of linear factors. What are the roots of equation $p(x) = 0$?

Solution: By Theorem 2.5, $2-i$ is also a root, so $(x-2-i)(x-2+i) = x^2-4x+5$ is a factor of $p(x)$. By long division we obtain that $p(x) = (x^2 - 4x + 5)(x^2 + 4)$, so

$$p(x) = (x - 2 - i)(x - 2 + i)(x - 2i)(x + 2i)$$

and the roots of $p(x) = 0$ are $2 \pm i$ and $\pm 2i$.

8. In each case your response should refer by number to appropriate results in Section 2.2.1 as needed.

   (a) If a polynomial of degree n with real coefficients does not have n real zero (counting multiplicity) then it must have an irreducible quadratic factor. Justify this statement.

   (b) If r is a zero of polynomial $f(x)$ of multiplicity 5 and a zero of polynomial $g(x)$ of multiplicity 7, must it be a zero of polynomial $h(x) = f(x) + g(x)$? If so, can we determine it's multiplicity? If so what is it? If not, why not?

Solution: (a) The polynomial has n zero, by Theorem 2.4 (or FTA (II)). If it does not have n real zero, then at least one zero, say λ, is not real. By Theorem 2.5, $\overline{\lambda}$ is also a zero. By Theorem 2.2, $(x-\lambda)(x-\overline{\lambda})$ is a quadratic factor; it is irreducible because its linear factors are not real.

(b) According to Theroem 2.4 (or Theorem 2.2) and the definition of "multiplicity"we can write

$$f(x) = (x-r)^5 p(x), \text{ and} g(x) = (x-r)^7 q(x),$$

where $p(r) \neq 0$ and $q(r) \neq 0$. Thus,

$$f(x)+g(x) = (x-r)^5 p(x)+(x-r)^7 q(x) = (x-r)^5 \left(p(x) + (x-r)^2 q(x)\right),$$

so r is a zero of $f(x)+g(x)$ of multiplicity at least 5. However, r is not a zero of $p(x)+(x-r)^2 q(x)$ since $p(r)+(r-r)^2 q(r) = p(r) \neq 0$, so the multiplicity of this zero is can be determined: it is 5.