Mathematics 1210 Assignment 3 Fall 2010

1. Prove using mathematical induction that for any $n \geq 2$ and collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,
$$\det(A_1 A_2 \cdots A_n) = (\det A_1)(\det A_2)\cdots(\det A_n).$$

Solution: Let m be any positive integer, and take $A_1, A_2, \ldots, A_n, \ldots$ to be any series of $m \times m$ matrices.

For $n \geq 2$ let S_n be the statement that
$$\det(A_1 A_2 \cdots A_n) = (\det A_1)(\det A_2)\cdots(\det A_n).$$

By Theorem XXX, $\det(A_1 A_2) = (\det A_1)(\det A_2)$. Therefore S_2 holds.

Now suppose S_k holds. (Induction Hypothesis)

Then
$$\begin{aligned}
\det(A_1 A_2 \cdots A_{k+1}) &= \det((A_1 \cdots A_k) A_{k+1}) \\
&= \det(A_1 \cdots A_k) \det(A_{k+1}) && \text{(by Th. XXX)} \\
&= [(\det A_1)(\det A_2)\cdots(\det A_k)]\,(\det A_{k+1}) && \text{(by Ind. Hyp.)} \\
&= (\det A_1)(\det A_2)\cdots(\det A_{k+1}).
\end{aligned}$$

Therefore S_{k+1} holds. That is, S_k implies S_{k+1}.

By the Principle of Mathematical Induction we conclude that S_n holds for all $n \geq 2$.

2. Prove using mathematical induction that for any $n \geq 1$, the determinant of an upper-triangular $n \times n$ matrix is the product of its diagonal entries.

Solution: Let P_n be the statement that the determinant of any $n \times n$ upper-triangular matrix is the product of its diagonal entries.

Consider the case $n = 1$. Since $\det(a)_{1\times 1} = a$ for any $a \in \mathbb{R}$, P_1 is true.

Now suppose P_k is true. Let $A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_{k+1} \end{pmatrix}$ (i.e., A is an arbitrary $(k+1) \times (k+1)$ matrix).

By cofactor expansion along the $(k+1)^{\text{st}}$ row,

$$\begin{aligned}
\det A &= 0 + 0 + \cdots + 0 + (-1)^{(k+1)+(k+1)} a_{k+1} \det \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_k \end{pmatrix} \\
&= a_{k+1} \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_k \end{pmatrix} \\
&= a_{k+1}(a_1 \ldots a_k) && \text{(by Ind. Hyp)} \\
&= a_1 a_2 \cdots a_{k+1}
\end{aligned}$$

Thus P_{k+1} is true. That is, P_k implies P_{k+1}.

Therefore, by the Principle of Mathematical Induction P_n holds, for all $n \geq 1$.

3. Is it true that for any two matrices A and B,

$$\det(A+B) = \det(A) + \det(B)?$$

If so, prove it. If not, find a counterexample.

Solution: This is not true in general, as seen by taking

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det(A+B) &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\ \det(A) + \det(B) &= 0 + 0 = 0 \qquad\qquad \text{so,} \\ \det(A+B) &\neq \det(A) + \det(B). \end{aligned}$$

4. Solve the following system using **the strict textbook version of Gaussian Elimination:**

$$\begin{array}{ccccccc} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x^2 & + & x_3 & = & -4 \end{array}$$

Solution: The corresponding augmented matrix reduces as follows:

$$\left(\begin{array}{ccc|c} 0 & 2 & -1 & -1 \\ 3 & -2 & 1 & 4 \\ 3 & 2 & 1 & -4 \end{array}\right) \begin{array}{c} R_2 \\ R_1 \\ \end{array} = \left(\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 0 & 2 & -1 & -1 \\ 3 & 2 & 1 & -4 \end{array}\right) \begin{array}{c} \frac{1}{3}R_1 \\ \\ \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 2 & -1 & -1 \\ 3 & 2 & 1 & -4 \end{array}\right) \begin{array}{c} \\ \\ R_3 - 3R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 2 & -1 & -1 \\ 0 & 4 & 0 & -8 \end{array}\right) \begin{array}{c} \\ \frac{1}{2}R_2 \\ \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 4 & 0 & -8 \end{array}\right) \begin{array}{c} \\ \\ R_3 - 4R_2 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 2 & -6 \end{array}\right) \begin{array}{c} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Which is in RREF. Back-substituting we obtain, in turn,

$$\begin{aligned} x_3 &= -3 \\ x_2 &= -\frac{1}{2} + \frac{1}{2}x_3 = -\frac{1}{2} - \frac{3}{2} = -2 \\ x_1 &= \frac{4}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_3 = \frac{4}{3} - \frac{4}{3} + \frac{3}{3} = 1 \end{aligned}$$

So the solution is $[x_1, x_2, x_3] = [1, -2, -3]$.

5. Solve the following system by putting the augmented matrix into RREF:

$$\begin{array}{ccccccc} x_1 & - & 3x_2 & & & = & -5 \\ & & x_2 & + & 3x_3 & = & -1 \\ 2x_1 & - & 10x_2 & + & 2x_3 & = & -20 \end{array}$$

Solution: One series of valid EROs that performs that performs the required reduction:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 2 & -10 & 2 & -20 \end{array}\right) \; R_3 - 2R_1 & = \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & -4 & 2 & -10 \end{array}\right) \; R_3 + 4R_2 \\ & = \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 14 & -14 \end{array}\right) \; \tfrac{1}{14}R_3 \\ & = \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \; R_2 - 3R_3 \\ & = \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \; R_1 + 3R_2 \\ & = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \end{aligned}$$

So the solution is $[x_1, x_2, x_3] = [1, 2, -1]$.

6. Solve the following system using Cramer's Rule:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & -1 \\ & & -x_2 & + & 2x_3 & = & -9 \\ 2x_1 & + & x_2 & & & = & 15 \end{array}$$

Solution: The coefficient matrix is $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$. Replacing each of the columns by the column of constant coefficients we obtain, respectively, $A_1 = \begin{pmatrix} -1 & 0 & 3 \\ -9 & -1 & 2 \\ 15 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & -1 & 3 \\ 0 & -9 & 2 \\ 2 & 15 & 0 \end{pmatrix}$, and $A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -9 \\ 2 & 1 & 15 \end{pmatrix}$.

The determinants of each of these are

$\det A = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 1 & -6 \end{vmatrix} = 4,$

$\det A_1 = \begin{vmatrix} -1 & 0 & 3 \\ -9 & -1 & 2 \\ 15 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -9 & -1 & -25 \\ 15 & 1 & 45 \end{vmatrix} = -(-45+25) = 20,$

$\det A_2 = \begin{vmatrix} 1 & -1 & 3 \\ 0 & -9 & 2 \\ 2 & 15 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 0 & -9 & 2 \\ 0 & 17 & -6 \end{vmatrix} = 54 - 34 = 20,$ and

$\det A_3 = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & -9 \\ 2 & 1 & 15 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & -9 \\ 0 & 1 & 17 \end{vmatrix} = -17 + 9 = -8.$

So by Cramer's Rule the solution is

$[x_1, x_2, x_3] = \left[\frac{\det A_1}{\det A}, \frac{\det A_2}{\det A}, \frac{\det A_3}{\det A}\right] = \left[\frac{20}{4}, \frac{20}{4}, \frac{-8}{4}\right] = [5, 5, -2].$

7. Prove the following property: for all $a, b, c \in \mathbb{R}$, $a, b, c \neq 0$,

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Solution: Any valid method of finding determinants is acceptable, but sufficiently many steps must be shown to justify the conclusion. Here we combine row/column operations and cofactor expansions one at a time so that steps should be obvious (when reducing *matrices* EROs must be annotated, whereas when using EROs and ECOs to find *determinants* we require only that they not be combined in a fashion so as to obscure the order of steps).

$$\begin{aligned} \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} &= \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ 1-(1+a)(1+c) & 1-(1+c) & 0 \end{vmatrix} \\ &= 1 \cdot \begin{vmatrix} -a & b \\ -a-c-ac & -c \end{vmatrix} - 0 + 0 = (-a)(-c) - (b)(-a-c-ac) \\ &= ac + ab + bc + abc = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \end{aligned}$$

8. Let $\mathbf{x}_0$ and $\mathbf{x}_1$ be solutions to the system of equations $A\mathbf{x} = \mathbf{b}$. Prove then that $\mathbf{x}_0 - \mathbf{x}_1$ is a solution to the (corresponding) homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

Solution: By assumption, $A\mathbf{x}_0 = A\mathbf{x}_1 = \mathbf{b}$. Taking $\mathbf{x} = \mathbf{x}_0 - \mathbf{x}_1$ we have

$$A\mathbf{x} = A(\mathbf{x}_0 - \mathbf{x}_1) = A\mathbf{x}_0 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

as required.

9. (a) Let $c \in \mathbb{R}$. Prove using mathematical induction that for any $n \geq 1$ and any $n \times n$ matrix A, $|cA| = c^n|A|$.

   (b) A square matrix is called **skew-symmetric** if $A^\top = -A$. Use part (a) and a property of determinants when taking transposes to show that every skew-symmetric 1001×1001 matrix has determinant 0.

<u>**Solution:**</u> (a) There are two obvious approaches: (i) expand by cofactors (for the inductive step) or (ii) Prove a different statement by induction, from which this follows, namely that, for an $n \times n$ matrix A if A_k is obtained by multiplying the first k rows of A by c then $|A_k| = c^k|A|$. The latter is a bit awkward as an induction proof, so we do the former.

Let P_n be the statement that for any $n \times n$ matrix A, $|cA| = c^n|A|$.

Consider the case $n = 1$: $\det(c(a)) = \det(ca) = c^1 a$, so P_1 holds.

Now suppose P_k holds. Suppose $M = [m_{ij}]_{(k+1)\times(k+1)}$. Let C_{ij} be the (i, j)-cofactor of M. Since C_{ij} is the determinant of a $k \times k$ submatrix of M, by our inductive hypothesis, the corresponding cofactor of cM is $c^k C_{ij}$. Therefore, by cofactor expansion along the first row we have that

$$|cM| = (cm_{11})(c^k C_{11}) + \cdots + (cm_{1,k+1})(c^k C_{1,k+1}) = c^{k+1}(m_{11}C_{11} + \cdots + m_{1,k+1}C_{1,k+1}) = c^{k+1}|M|.$$

So P_{k+1} holds. That is, P_k implies P_{k+1}.

It follows that P_n holds for all $n \geq 1$.

(b) If A is a 1001×1001 matrix then by Theorem XXX we have $|A^\top| = |A|$ but by part (a) we also have $|A^\top| = |-A| = |(-1)A| = (-1)^{1001}|A| = -|A|$. So $|A| = -|A|$. It follows that $|A| = 0$.

10. An **elementary matrix** is a matrix which is one elementary row operation away from the identity matrix. For instance,

$$E_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

are all elementary matrices.

(a) Let k be any real number, $k \neq 0$. Find an elementary matrix with determinant k.

(b) **BONUS: 3 MARKS**. Let E be an $n \times n$ elementary matrix formed by performing row operation r to the identity I_n Let A be any $n \times n$ matrix. Then the matrix product EA will result in the matrix obtained by peforming r to A. Use this fact, and properties of determinants to formally prove the following theorem: If A is an $n \times n$ matrix such that the reduced row echelon form of A is I_n, then $\det(A) \neq 0$.

Solution: (a) $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

(b) From the givens in the statement of the question, A is reduced to I_n by a finite series of EROs, and the effect of each ERO is to premultiply the matrix upon which it acts by an elementary matrix. Therefore there exist elementary matrices $E_1, E_2, \ldots, E_m$ such that

$$E_m E_{m-1} \cdots E_2 E_1 A = I_n.$$

Performing determinants on both sides and using the multiplicative property of determinants, we have

$$|E_m E_{m-1} \cdots E_2 E_1 A| = |I_n| = 1 = |E_m E_{m-1} \cdots E_2 E_1| \cdot |A|,$$

which is impossible if $\det A = 0$. Therefore, $\det A \neq 0$.