Mathematics 1210 Assignment 5 Key Fall 2010

1. Use the direct method to find the inverse of

$$\begin{pmatrix} 2 & 1 & -1 \\ -13 & -12 & 13 \\ 10 & 10 & -11 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -13 & -12 & 13 & 0 & 1 & 0 \\ 10 & 10 & -11 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} R_2 + 7R_1 \\ R3 - 5R_1 \end{array} \equiv \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & -5 & 6 & 7 & 1 & 0 \\ 0 & 5 & -6 & -5 & 0 & 1 \end{array}\right) \; R_2 + R_3$$

$$\equiv \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 5 & -6 & -5 & 0 & 1 \end{array}\right) \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 0 & 1 & -1 & -3 & -2 & -2 \\ 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 5 & -6 & -5 & 0 & 1 \end{array}\right) \begin{array}{c} R_2 \\ R_1 \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & -1 & -3 & -2 & -2 \\ 0 & 5 & -6 & -5 & 0 & 1 \end{array}\right) \begin{array}{c} \\ \\ R_3 - 5R_2 \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & -1 & -3 & -2 & -2 \\ 0 & 0 & -1 & 10 & 10 & 11 \end{array}\right) \begin{array}{c} \\ R_2 - R_3 \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -13 & -12 & -13 \\ 0 & 0 & -1 & 10 & 10 & 11 \end{array}\right) \begin{array}{c} \\ \\ -R_3 \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -13 & -12 & -13 \\ 0 & 0 & 1 & -10 & -10 & -11 \end{array}\right)$$

Therefore, $A^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ -13 & -12 & -13 \\ -10 & -10 & -11 \end{pmatrix}$.

2. Find the adjoint of
$$\begin{pmatrix} 6 & 10 & 5 \\ -3 & -5 & -3 \\ -7 & -11 & -5 \end{pmatrix}$$

Solution:

$$\begin{aligned}
\operatorname{adj}\begin{pmatrix} 6 & 10 & 5 \\ -3 & -5 & -3 \\ -7 & -11 & -5 \end{pmatrix} &= \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}^{\top} \\
&= \begin{pmatrix} (25-33) & -(15-21) & (33-35) \\ -(-50+55) & (-30+35) & -(-66+70) \\ (-30+25) & -(-18+15) & (-30+30) \end{pmatrix} \\
&= \begin{pmatrix} -8 & -5 & -5 \\ 6 & 5 & 3 \\ -2 & -4 & 0 \end{pmatrix}
\end{aligned}$$

3. Use the adjoint method to find the inverse of

$$\begin{pmatrix} 0 & 3 & 5 & 4 \\ 7 & 0 & 2 & 0 \\ 10 & 0 & 3 & 0 \\ 0 & 1 & 9 & 0 \end{pmatrix}$$

Solution: Strategically expanding by minors we obtain

$$\begin{vmatrix} 0 & 3 & 5 & 4 \\ 7 & 0 & 2 & 0 \\ 10 & 0 & 3 & 0 \\ 0 & 1 & 9 & 0 \end{vmatrix} = -4 \begin{vmatrix} 7 & 0 & 2 \\ 10 & 0 & 3 \\ 0 & 1 & 9 \end{vmatrix} = (-4)(-1) \begin{vmatrix} 7 & 2 \\ 10 & 3 \end{vmatrix} = 4(21 - 20) = 4.$$

Now,

$$\begin{aligned} \operatorname{adj} \begin{pmatrix} 0 & 3 & 5 & 4 \\ 7 & 0 & 2 & 0 \\ 10 & 0 & 3 & 0 \\ 0 & 1 & 9 & 0 \end{pmatrix} &= \begin{pmatrix} M_{11} & -M_{12} & M_{13} & -M_{14} \\ -M_{21} & M_{22} & -M_{23} & M_{24} \\ M_{31} & -M_{32} & M_{33} & -M_{34} \\ -M_{41} & M_{42} & -M_{43} & M_{44} \end{pmatrix}^{\top} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 12 & 90 & -40 & -220 \\ -8 & -252 & 28 & 154 \\ 0 & 4 & 0 & -3 \end{pmatrix}^{\top} \\ &= \begin{pmatrix} 0 & 12 & -8 & 0 \\ 0 & 90 & -252 & 4 \\ 0 & -40 & 28 & 0 \\ 1 & -220 & 154 & -3 \end{pmatrix}. \end{aligned}$$

So

$$\begin{pmatrix} 0 & 3 & 5 & 4 \\ 7 & 0 & 2 & 0 \\ 10 & 0 & 3 & 0 \\ 0 & 1 & 9 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & 12 & -8 & 0 \\ 0 & 90 & -252 & 4 \\ 0 & -40 & 28 & 0 \\ 1 & -220 & 154 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 & 0 \\ 0 & \frac{45}{2} & -63 & 1 \\ 0 & -10 & 7 & 0 \\ \frac{1}{4} & -55 & \frac{77}{2} & -\frac{3}{4} \end{pmatrix}.$$

4. (a) Find the inverse of $\begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix}$.

(b) Use part (a) to solve the system

$$\begin{array}{rcrcr} 6x & + & 5y & = & 3 \\ 9x & + & 7y & = & -2. \end{array}$$

(c) Let $a, b \in \mathbb{R}$. Use part (a) to solve the system

$$\begin{array}{rcrcr} 6x & + & 5y & = & a \\ 9x & + & 7y & = & b. \end{array}$$

(d) Use part (a) to find the solution to the matrix equation

$$\begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Solution: (a) $\begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix}^{-1} = \frac{1}{42-45} \begin{pmatrix} 7 & -5 \\ -9 & 6 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -7 & 5 \\ 9 & -6 \end{pmatrix}$.

(b) In matrix form the system is $\begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, which has solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -7 & 5 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{-31}{3} \\ \frac{39}{3} \end{pmatrix}.$$

So $x = -\frac{31}{3}$ and $y = 13$.

(c) Similarly in matrix form this system is $\begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, which has solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -7 & 5 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5b - 7a \\ 9a - 6b \end{pmatrix}.$$

So $x = \frac{5b-7a}{3}$ and $y = 3a - 2b$.

(d)

$$X = \begin{pmatrix} 6 & 5 \\ 9 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -7 & 5 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 8 & 6 \\ -9 & -6 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} & 2 \\ 3 & -2 \end{pmatrix}.$$

5. Let $V = \{[5,-7,-1,2],[1,1,2,6],[0,2,4,6]\}$.

   (a) Express $[4,-4,5,8]$ as a linear combination of the vectors in V.

   (b) Prove $[1,0,0,0]$ cannot be expressed as a linear combination of the vectors in V.

Solution: (a) Solving for the coefficients in

$$c_1[5,-7,-1,2]+c_2[1,1,2,6]+c_3[0,2,4,6]=[4,-4,5,8]$$

is equivalent to solving the system whose augmented matrix reduces as follows:

$$\left(\begin{array}{rrr|r} 5&1&0&4\\ -7&1&2&-4\\ -1&2&4&5\\ 2&6&6&8\end{array}\right)\begin{array}{l} R_1+5R_3\\ R_2-7R_3\\ \\ R_4+2R_3\end{array} \equiv \left(\begin{array}{rrr|r} 0&11&20&29\\ 0&-13&-26&-39\\ -1&2&4&5\\ 0&10&14&18\end{array}\right)\begin{array}{l} R_1-R_4\\ R_2+R_4\\ \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{rrr|r} 0&1&6&11\\ 0&-3&-12&-21\\ -1&2&4&5\\ 0&10&14&18\end{array}\right)\begin{array}{r} \\ R_2+3R_1\\ R_3-2R_1\\ R_4-10R_1\end{array}$$

$$\equiv \left(\begin{array}{rrr|r} 0&1&6&11\\ 0&0&6&12\\ -1&0&-8&-17\\ 0&0&-46&92\end{array}\right)\begin{array}{l} \\ \frac{1}{6}R_2\\ \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{rrr|r} 0&1&6&11\\ 0&0&1&2\\ -1&0&-8&-17\\ 0&0&-46&92\end{array}\right)\begin{array}{r} R_1-6R_2\\ \\ R_3+8R_2\\ R_4+48R_2\end{array}$$

$$\equiv \left(\begin{array}{rrr|r} 0&1&0&-1\\ 0&0&1&2\\ -1&0&0&-1\\ 0&0&0&0\end{array}\right)\begin{array}{r} -R_3\\ R_1\\ R_2\\ \\ \end{array}$$

$$\equiv \left(\begin{array}{rrr|r} 1&0&0&1\\ 0&1&0&-1\\ 0&0&1&2\\ 0&0&0&0\end{array}\right),$$

so $[4,-4,5,8]=[5,-7,-1,2]-[1,1,2,6]+2[0,2,4,6]$.

(b) Replacing $[4,-4,5,8]$ with $[1,0,0,0]$ and performing the same series of steps above we obtain that

$$\left(\begin{array}{rrr|r} 5&1&0&1\\ -7&1&2&0\\ -1&2&4&0\\ 2&6&6&0\end{array}\right) \equiv \cdots \equiv \left(\begin{array}{rrr|r} 1&0&0&-2\\ 0&1&0&-2\\ 0&0&1&\frac{1}{2}\\ 0&0&0&14\end{array}\right),$$

So $c_1[5,-7,-1,2]+c_2[1,1,2,6]+c_3[0,2,4,6]=[1,0,0,0]$ has no solution. It follows that the linear combination in question does not exist.

6. Show that **every** 3-tuple (vector in $\mathbb{E}^3$) can be expressed as a linear combination of the vectors $[1,2,3],[4,5,0],[6,0,0]$.

Solution: It suffices to show *any one* of the following:
(i) the matrix A with columns $[1,2,3],[4,5,0],[6,0,0]$ has rank 3;
(ii) A has nonzero determinant;
(iii) A is row-equivalent to I_3; or
(iv) the system $x[1,2,3]+y[4,5,0]+z[6,0,0]=[a,b,c]$ has a solution for every triple $(a,b,c)\in \mathbb{R}^3$; or (iv) $x[1,2,3]+y[4,5,0]+z[6,0,0]=[0,0,0]$ has no nontrivial solutions.
(All these statements are equivalent, by various results in the course, in the case of n vectors in $\mathbb{E}^n$.

We opt for (ii), which is simplest here:

$$\det\left(\begin{array}{c|c|c} 1 & 4 & 6 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{array}\right) = -\begin{vmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & 4 & 6 \end{vmatrix} = -90 \neq 0.$$

Since the matrix has nonzero determinant the system described in (iv) always has a solution, so **every** 3-tuple can be expressed as a linear combination of the given vectors.

7. (a) Show that $\{[-5,6,-4,-6],[-10,4,7,3],[9,4,3,7]\}$ is linearly independent.

   (b) Show that $\{[1,9,-4,-10],[-4,3,-2,1],[14,-35,0,14]\}$ is linearly **independent** (note fixed typo from original version of problem).

   (c) Show that $\{[2,-5,0,2],[-4,10,0,-4],[-6,-4,-3,1]\}$ is linearly dependent.

Solution: In each case, we determine the rank of the matrix whose columns are the given vectors. The vectors are linearly independent if and only if the rank of the matrix is equal to the number of vectors (equivalently, in REF there is a leading 1 in each column). As discussed in class this is equivalent to obtaining that the system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots = \mathbf{0}$ has only the trivial solution $c_1 = c_2 = \cdots = 0$; in other words, the vectors are linearly independent.

(a)
$$\left(\begin{array}{c|c|c} -5 & -10 & 9 \\ 6 & 4 & 4 \\ -4 & 7 & 3 \\ -6 & 3 & 7 \end{array}\right) \begin{array}{l} R_1+R_2 \\ \\ R_3+R_2 \\ R_4+R_2 \end{array} \equiv \begin{pmatrix} 1 & -6 & 13 \\ 6 & 4 & 4 \\ 2 & 11 & 7 \\ 0 & 7 & 11 \end{pmatrix} \begin{array}{l} \\ R_2-6R_1 \\ R_3-2R_1 \\ \\ \end{array}$$
$$\equiv \begin{pmatrix} 1 & -6 & 13 \\ 0 & 40 & -74 \\ 0 & 23 & -19 \\ 0 & 7 & 11 \end{pmatrix} \begin{array}{l} \\ R_2-6R_4 \\ R_3-3R_4 \\ \\ \end{array} \equiv \begin{pmatrix} 1 & -6 & 13 \\ 0 & -2 & -140 \\ 0 & 2 & -52 \\ 0 & 7 & 11 \end{pmatrix} \begin{array}{l} \\ \\ R_3+R_2 \\ R_4+\frac{7}{2}R_2 \end{array}$$
$$\equiv \begin{pmatrix} 1 & -6 & 13 \\ 0 & -2 & -140 \\ 0 & 0 & -192 \\ 0 & 0 & -479 \end{pmatrix} \equiv \cdots \equiv \begin{pmatrix} 1 & -6 & 13 \\ 0 & 1 & 70 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
which has rank 3, so the set of vectors is linearly independent.

(b)
$$\left(\begin{array}{c|c|c} 1 & -4 & 14 \\ 9 & 3 & -35 \\ -4 & -2 & 0 \\ -10 & 1 & 14 \end{array}\right) \begin{array}{l} \\ R_2-9R_1 \\ R_3+4R_1 \\ R_4+10R_1 \end{array} \equiv \begin{pmatrix} 1 & -4 & 14 \\ 0 & 39 & -161 \\ 0 & -18 & 56 \\ 0 & -39 & 154 \end{pmatrix} \begin{array}{l} \\ \\ \\ R_4+R_2 \end{array}$$
$$\equiv \begin{pmatrix} 1 & -4 & 14 \\ 0 & 39 & -161 \\ 0 & -18 & 56 \\ 0 & 0 & -7 \end{pmatrix} \equiv \cdots \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
which has rank 3, so the set is linearly independent.

(c)
$$\left(\begin{array}{c|c|c} 2 & -4 & -6 \\ -5 & 10 & -4 \\ 0 & 0 & -3 \\ 2 & -4 & 1 \end{array}\right) \begin{array}{l} \\ R_2+3R_1 \\ \\ R_4-R_1 \end{array} \equiv \begin{pmatrix} 2 & -4 & -6 \\ 1 & -2 & -16 \\ 0 & 0 & -3 \\ 0 & 0 & 7 \end{pmatrix} \begin{array}{l} R_1-2R_2 \\ \\ \\ \\ \end{array}$$
$$\equiv \begin{pmatrix} 0 & 0 & 26 \\ 1 & -2 & -16 \\ 0 & 0 & -3 \\ 0 & 0 & 7 \end{pmatrix} \equiv \cdots \equiv \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
which has rank 2, so the set is linearly dependent.

8. Let $V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a linearly independent set of vectors. Prove (using the definition of linearly independent) that if we remove the first vector in the set, the remaining set is still linearly independent

Solution: We use the second definition of *linearly independent*: the only linear combination of the set equal to zero is the trivial (or zero) linear combination.

Suppose $\{\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is linearly dependent. Then

$$c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$

with some $c_i \neq 0$. Then

$$0\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

is also a nontrivial linear combination, and so $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent.

It follows that, if the original set is linearly independent, then so is $\{\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$.

9. Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be some set of vectors in $\mathbb{E}^m$. Assume that $\mathbf{w}_1$ and $\mathbf{w}_2$ are two vectors that can be written as linear combinations of the vectors in V. Prove then that the vector $\mathbf{w}_1 + \mathbf{w}_2$ can also be written as a linear combination of the vectors in V.

Solution: Suppose that

$$\mathbf{w}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

and

$$\mathbf{w}_2 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n.$$

Then

$$\begin{aligned} \mathbf{w}_1 + \mathbf{w}_2 &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n) \\ &= (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n, \end{aligned}$$

so $\mathbf{w}_1 + \mathbf{w}_2$ is a linear combination of the vectors in V (having coefficients $a_1 + b_1, \ldots, a_n + b_n$, respectively), as required.

10. For which values of a can every vector in $\mathbb{E}^3$ be expressed as a linear combination of the following vectors:

$$[a, 1, 1], [1, 1, a], [1, a, 1]?$$

Solution: Let (x, y, z) be an arbitrary vector in $\mathbb{E}^3$. Let c_1, c_2, c_3 be such that

$$c_1[a, 1, 1] + c_2[1, 1, a] + c_3[1, a, 1] = [x, y, z]$$

$$\begin{aligned} c_1[a, 1, 1] + c_2[1, 1, a] + c_3[1, a, 1] &= [x, y, z] \\ [ac_1, c_1, c_1] + [c_2, c_2, ac_2] + [c_3, ac_3, c_3] &= [x, y, z] \\ [ac_1 + c_2 + c_3, c_1 + c_2 + ac_3, c_1 + ac_2 + c_3] &= [x, y, z]. \end{aligned}$$

So we have the system of equations

$$ac_1 + c_2 + c_3 = x, \qquad c_1 + c_2 + ac_3 = y, \qquad c_1 + ac_2 + c_3 = z,$$

or written as a matrix equation,

$$\begin{pmatrix} a & 1 & 1 \\ 1 & 1 & a \\ 1 & a & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We want this system to have a solution for all x, y, z. This will happen exactly when the coefficient matrix is invertible, which happens exactly when the determinant is not zero:

$$\begin{aligned} \begin{vmatrix} a & 1 & 1 \\ 1 & 1 & a \\ 1 & a & 1 \end{vmatrix} &= a \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} \\ &= a(1 - a^2) - (1 - a) + (a - 1) \\ &= a - a^3 - 1 + a + a - 1 \\ &= -a^3 + 3a - 2 \\ &= (a - 1)(-a^2 - a + 2) \\ &= -(a - 1)(a^2 + a - 2) \\ &= -(a - 1)(a + 2)(a - 1) \\ &= -(a - 1)^2(a + 2). \end{aligned}$$

Therefore, the coefficient matrix has determinant 0 exactly when $a = 1$ or $a = -2$, and thus the system has a solution when $a \neq 1$ or -2.

Alternatively one can reduce the question to finding when the above determinant is nonzero by considering Theorem 3.11. One can also do MORE work by solving the system explicitly instead of looking at the determinant. Or one could look at the rank of the matrix. Any valid approach along such lines is acceptable.