UNIVERSITY OF MANITOBA

DATE: October 20, 2010 MIDTERM EXAMINATION

 SOLUTIONS

DEPARTMENT & COURSE NO: MATH 1210 TIME: 1 hour

EXAMINATION: Techniques of Classical and Linear Algebra EXAMINER: Craigen/Borgersen

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[4] 1. Write the following sum using sigma notation with an index starting at 1:

$$n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 + \cdots + (-1)^{n-1}(1)^2.$$

$$\sum_{i=1}^{n} (-1)^{n-i} i^2$$

[2] 2. For each $n \geq 1$, the n-th triangular number is $T_n = 1 + 2 + \cdots + n$. Prove using Mathematical Induction that, for each $n \geq 2$,

$$T_n + T_{n-1} = n^2.$$

No marks will be awarded for any other method.

Let P_n be $T_n + T_{n-1} = n^2$.

Then the base case is P_2: $T_2 + T_1 = 2^2 = 4$.

$LS = (1+2) + 1 = 4 = RS$.

Assume P_k: $T_k + T_{k-1} = k^2$

Prove P_{k+1}: $T_{k+1} + T_k = (k+1)^2$

$LS = T_{k+1} + T_k = T_{k-1} + k + k+1 + T_k$

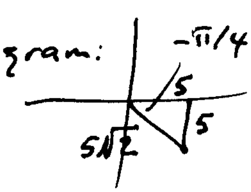
$= k^2 + 2k + 1$ by assumption

$= (k+1)^2 = RS$.

By PMI, P_n is true for all $n \geq 2$.

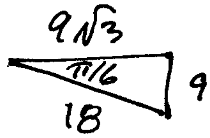
[3] 3. (a) Write the following complex number in Cartesian form: $5\sqrt{2}e^{\frac{15\pi}{4}i}$

This can be done with algebra or a good diagram:



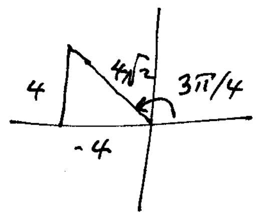
The number is $5 - 5i$.

[4] (b) Write $9\sqrt{3} - 9i$ in polar form.



The number is $18\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

[7] (c) Find the cube roots of $-4 + 4i$. Express your answers in exponential form.



$$-4+4i = 4\sqrt{2}\, e^{3\pi i/4} = 2^{\frac{5}{2}} e^{3\pi i/4}$$

$$= 2^{\frac{5}{2}} e^{11\pi i/4} \longrightarrow 2^{5/6} e^{11\pi i/12}$$

$$= 2^{\frac{5}{2}} e^{19\pi i/4} \longrightarrow 2^{5/6} e^{19\pi i/12}$$

$$= 2^{5/2} e^{27\pi i/4} \longrightarrow 2^{5/6} e^{27\pi i/12}$$

$$= 2^{5/6} e^{3\pi i/12}$$

4. Consider the polynomial $f(x) = 6x^5 - 3x^4 - 5x^2 + x + 4$. Without factoring this polynomial or attempting to find any zeros for it:

[4] (a) Use the Rational Root Theorem to write down all possible rational zeros of $f(x)$.

If p/q is a rational root (in lowest terms), then

$p \mid 4$, $q \mid 6$ $\Rightarrow p \in \{\pm1, \pm2, \pm4\}$, $q \in \{\pm1, \pm2, \pm3, \pm6\}$

$\Rightarrow p/q \in \{\pm1, \pm2, \pm4, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{1}{6}\}$

[3] (b) Use Descartes' Rules of Signs to determine how many positive real zeros $f(x)$ may have and how many negative real zeros $f(x)$ may have.

$f(x) = 6x^5 - 3x^4 - 5x^2 + x + 4 \Rightarrow$ 2 sign changes $\Rightarrow$ 2 or 0 +ve real roots

$f(-x) = -6x^5 - 3x^4 - 5x^2 - x + 4 \Rightarrow$ 1 sign change $\Rightarrow$ 1 −ve real root

[3] (c) Use the Bounds Theorem to determine how large the absolute value of a root of $f(x)$ may be.

If x is any root of $f(x)=0$, then

$$|x| < \frac{\max(|-3|, |-5|, |1|, |4|)}{|6|} + 1 = \frac{5}{6} + 1 = \frac{11}{6}$$

[3] (d) Use your answers to (b) and (c) to improve your answer to (a) (that is, taking in account your answers for (b) and (c), list all possible rational zeros of $f(x)$).

$$\frac{p}{q} \in \left\{ \pm 1, \pm \tfrac{1}{2}, \pm \tfrac{1}{3}, \pm \tfrac{2}{3}, \pm \tfrac{4}{3}, \pm \tfrac{1}{6} \right\}$$

[3] (e) If $a+bi$ is a complex root of $f(x)$, where $b \neq 0$, find another root, and write down an irreducible real quadratic factor of $f(x)$.

$$f(a+bi)=0 \Rightarrow f(a-bi)=0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x-(a+bi) \mid f(x) \qquad x-(a-bi) \mid f(x)$$

$$\Rightarrow (x-a-bi)(x-a+bi) = x^2 - ax + \cancel{bix} - ax + a^2 - \cancel{abi} - \cancel{bix} + \cancel{abi} - b^2 i^2$$

$$= \underline{x^2 + (-2a)x + (a^2+b^2)}$$

is an irreducible real quadratic factor of $f(x)$.

5. Let $P = (1, -1, 0)$, $\vec{u} = [1, 2, -3]$ and $\vec{v} = [2, 0, 2]$.

[3] (a) Find $\cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

$$\underset{\sim}{u} \circ \underset{\sim}{v} = (1,2,-3)\circ(2,0,2) = 2+0-6 = -4$$

$$\|\underset{\sim}{u}\| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$$

$$\|v\| = \sqrt{2^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\cos\theta = \frac{\underset{\sim}{u}\circ \underset{\sim}{v}}{\|\underset{\sim}{u}\|\cdot\|\underset{\sim}{v}\|} = \frac{-4}{\sqrt{14}\sqrt{8}} = \frac{-4}{\sqrt{14}\cdot 2\sqrt{2}} = \frac{-2}{\sqrt{14}\sqrt{2}} = \frac{-2}{\cancel{\sqrt{2}}\sqrt{7}\cancel{\sqrt{2}}} = \frac{-1}{\sqrt{7}} = \underline{\frac{-\sqrt{7}}{7}}$$

[3] (b) Give an equation in vector form for the line passing through P and with direction vector $\vec{v}$.

$$\underset{\sim}{X} = (1,-1,0) + t(2,0,2), \quad t \in \mathbb{R}$$

$$(x,y,z) = (1,-1,0) + t(2,0,2)$$

6. Let $\vec{x} = [1, 0, -4]$, $\vec{y} = [a, 1, 3]$.

[3] (a) Find all values of a such that $\vec{x}$ and $\vec{y}$ are perpendicular.

$$x \cdot y = a - 12 = 0$$

$$\Longleftrightarrow \underline{a = 12}$$

[5] (b) Find all values of a such that $||\vec{x} + 2\vec{y}|| = 3$.

$$||(1,0,-4) + 2(a,1,3)|| = ||(1+2a, 2, -4+6)|| = ||(1+2a, 2, 2)||$$

$$= \sqrt{(1+2a)^2 + 2^2 + 2^2} = \sqrt{1 + 4a + 4a^2 + 8}$$

$$= \sqrt{4a^2 + 4a + 9} = 3$$

$$\Longleftrightarrow 4a^2 + 4a + 9 = 9$$

$$4a^2 + 4a = 0$$

$$4a(a+1) = 0$$

$$\underline{a = 0} \text{ or } \underline{a = -1}.$$