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[30] **1.** Evaluate and simplify to Cartesian form:

(a) $i^{51}(\overline{7i-3})$

Solution: $i^{51}(\overline{7i-3}) = i^{48} \cdot i^3(-7i-3) = (i^4)^{12}(-i)(-7i-3) = -7+3i$

(b) $\dfrac{3i+4}{2+5i}$

Solution: $\dfrac{3i+4}{2+5i} = \dfrac{3i+4}{2+5i} \cdot \dfrac{2-5i}{2-5i} = \dfrac{6i+8+15-20i}{2^2+5^2} = \dfrac{23}{29} - \dfrac{14}{29}i$

(c) $\left(\dfrac{1+i}{i-\sqrt{3}}\right)^9$

Solution:
$$\left(\frac{1+i}{i-\sqrt{3}}\right)^9 = \left[\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)}{2\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)}\right]^9 = \left[\frac{\sqrt{2}e^{\frac{\pi}{4}i}}{2e^{\frac{5\pi}{6}i}}\right]^9 = \left[\frac{1}{\sqrt{2}}e^{-\frac{7\pi}{12}i}\right]^9$$
$$= \frac{1}{2^{9/2}}e^{-\frac{21\pi}{4}i} = \frac{1}{2^4\sqrt{2}}\left[\cos\left(-\frac{21\pi}{4}\right) + i\sin\left(-\frac{21\pi}{4}\right)\right]$$
$$= \frac{1}{16\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -\frac{1}{32} + \frac{1}{32}i$$

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[10] **2.** Express the sum

$$\frac{1}{5}-\frac{2}{7}+\frac{4}{9}-\frac{8}{11}+\frac{16}{13}-\frac{32}{15}+\frac{64}{17}-\frac{128}{19}+\frac{256}{21}-\frac{512}{23}+\frac{1024}{25}$$

in sigma notation.

Solution 1: $\displaystyle\sum_{k=0}^{10}(-1)^k\frac{2^k}{2k+5}$ Solution 2: $\displaystyle\sum_{k=1}^{11}(-1)^{k-1}\frac{2^{k-1}}{2k+3}$

Other solutions are possible.

[18] **3.** Evaluate the sum

$$\sum_{k=11}^{22}[5(k-4)^3+7]$$

using any of the following identities that you may find relevant:

$$\sum_{j=1}^{n} j=\frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2=\frac{n(n+1)(2n+1)}{6}, \qquad \sum_{j=1}^{n} j^3=\frac{n^2(n+1)^2}{4}.$$

Solution 1: Using the change of variables, $j=k-4$,

$$\begin{aligned}\sum_{k=11}^{22}[5(k-4)^3+7] &= \sum_{j=7}^{18}(5j^3+7)=5\sum_{j=7}^{18}j^3+7\sum_{j=7}^{18}1\\ &=5\left(\sum_{j=1}^{18}j^3-\sum_{j=1}^{6}j^3\right)+7\cdot 12=5\left(\frac{18^2\cdot 19^2}{4}-\frac{6^2\cdot 7^2}{4}\right)+84\\ &=5\cdot\frac{6^2}{4}(3^2\cdot 19^2-7^2)+84=5\cdot 9\cdot(3\cdot 19-7)(3\cdot 19+7)+84\\ &=5\cdot 9\cdot 50\cdot 64+84=1000\cdot 9\cdot 16+84=144084\end{aligned}$$

Solution 2: $\displaystyle\sum_{k=11}^{22}[5(k-4)^3+7]=\sum_{k=11}^{22}\left[5(k^3-12k^2+48k-64)+7\right]$

$$\begin{aligned}&=\sum_{k=11}^{22}(5k^3-60k^2+240k-313)=5\sum_{k=11}^{22}k^3-60\sum_{k=11}^{22}k^2+240\sum_{k=11}^{22}k-313\sum_{k=11}^{22}1\\ &=5\left(\sum_{k=1}^{22}k^3-\sum_{k=1}^{10}k^3\right)-60\left(\sum_{k=1}^{22}k^2-\sum_{k=1}^{10}k^2\right)+240\left(\sum_{k=1}^{22}k-\sum_{k=1}^{10}k\right)-313\cdot 12\\ &=5\left(\frac{22^2\cdot 23^2}{4}-\frac{10^2\cdot 11^2}{4}\right)-60\left(\frac{22\cdot 23\cdot 45}{6}-\frac{10\cdot 11\cdot 21}{6}\right)+240\left(\frac{22\cdot 23}{2}-\frac{10\cdot 11}{2}\right)-313\cdot 12\end{aligned}$$

[22] **4.** Find all solutions of the equation

$$z^6 + 3z^3 + 9 = 0\,.$$

Express your answers in exponential form.

Solution: Set $y = z^3$. Then $y^2 + 3y + 9 = 0$. Hence

$$y = \frac{-3 \pm \sqrt{3^2 - 4\cdot 9}}{2} = \frac{-3 \pm \sqrt{-27}}{2} = -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\,.$$

If $y = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$, then

$$z^3 = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 3e^{\frac{2\pi}{3}i}\,.$$

This leads to:

$$z = \left[3e^{\left(\frac{2\pi}{3}+2k\pi\right)i}\right]^{1/3} = \sqrt[3]{3}e^{\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)i}, \quad (k = 0, 1, 2).$$

Therefore the first three solutions are:

$$z = \sqrt[3]{3}e^{\frac{2\pi}{9}i},\ z = \sqrt[3]{3}e^{\frac{8\pi}{9}i}, \text{and } z = \sqrt[3]{3}e^{\frac{14\pi}{9}i} = \sqrt[3]{3}e^{-\frac{4\pi}{9}i}.$$

If $y = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$, then

$$z^3 = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i = 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 3e^{-\frac{2\pi}{3}i}\,.$$

This leads to:

$$z = \left[3e^{\left(-\frac{2\pi}{3}+2k\pi\right)i}\right]^{1/3} = \sqrt[3]{3}e^{\left(-\frac{2\pi}{9}+\frac{2k\pi}{3}\right)i}, \quad (k = 0, 1, 2).$$

Therefore there are three more solutions:

$$z = \sqrt[3]{3}e^{-\frac{2\pi}{9}i},\ z = \sqrt[3]{3}e^{\frac{4\pi}{9}i}, \text{and } z = \sqrt[3]{3}e^{\frac{10\pi}{9}i} = \sqrt[3]{3}e^{-\frac{8\pi}{9}i}\,.$$

In total, there are six solutions: $z = \sqrt[3]{3}e^{\pm\frac{2\pi}{9}i}$, $z = \sqrt[3]{3}e^{\pm\frac{4\pi}{9}i}$, and $z = \sqrt[3]{3}e^{\pm\frac{8\pi}{9}i}$.

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[20] **5.** Use mathematical induction to prove that

$$\sum_{j=1}^{n} \frac{1}{j} < \frac{2}{3}n$$

for every integer $n \geq 3$.

Solution: First, we show that the given inequality holds for $n = 3$:

$$\sum_{j=1}^{3} \frac{1}{j} < \frac{2}{3} \cdot 3\,.$$

Equivalently,

$$1 + \frac{1}{2} + \frac{1}{3} < 2.$$

This inequality holds since its left-hand side is equal to $\dfrac{11}{6}$.

Next, suppose that the given inequality holds for $n = k$, where $k \geq 3$:

$$\sum_{j=1}^{k} \frac{1}{j} < \frac{2}{3}k\,. \tag{1}$$

We will show that the given inequality must hold for $n = k + 1$:

$$\sum_{j=1}^{k+1} \frac{1}{j} < \frac{2}{3}(k+1)\,. \tag{2}$$

Starting from the left-hand side of (2), in view of the inductive hypothesis (1),

$$\sum_{j=1}^{k+1} \frac{1}{j} = \left(\sum_{j=1}^{k} \frac{1}{j}\right) + \frac{1}{k+1} < \frac{2}{3}k + \frac{1}{k+1}\,.$$

Therefore, in order to prove (2), it suffices to show that

$$\frac{2}{3}k + \frac{1}{k+1} < \frac{2}{3}(k+1)\,. \tag{3}$$

By subtracting $\dfrac{2}{3}k$ from both sides, the last inequality is simplified, and is equivalent to:

$$\frac{1}{k+1} < \frac{2}{3}\,.$$

Since this is equivalent to $k + 1 > \dfrac{3}{2}$, the inequality (3) obviously holds for $k \geq 3$. Thus (2) is proved.

Therefore, by the principle of mathematical induction, the given inequality holds for every integer $n \geq 3$.