UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS
MATH 1210 Techniques of Classical and Linear Algebra
SECOND TERM TEST
November 9, 2011 5:30 – 6:30 PM

LAST NAME: ______________________________

FIRST NAME: ______________________________

STUDENT NUMBER: ______________________________

SIGNATURE: ______________________________
(I understand that cheating is a serious offense)

Please indicate your instructor and section by checking the appropriate box below:

☐	A01	MWF (9:30 – 10:20 AM, EITC E3 270)	G. Krause
☐	A02	MWF (1:30 – 2:20 PM, St. Paul's College 305)	A. Prymak
☐	A03	MWF (1:30 – 2:20 PM, EITC E2 155)	M. Despic

INSTRUCTIONS TO STUDENTS:

Fill in all the information above.

This is a 1 hour exam.

***No** notes, books, cell phones, calculators or other computing devices are permitted.*

***Show your work clearly** for full marks.*

This test has a title page, 6 pages of questions, and 1 blank page for rough work. Please check that you have all pages.

The value of each question is indicated in the left-hand margin beside the statement of the question. The total value of all questions is 100.

*Answer all questions on the exam paper in the space provided. If you need more room, you may continue your work on the **reverse** side of the page, but **clearly indicate** that your work is continued there.*

Question:	1	2	3	4	5	6	Total
Points:	27	17	17	13	19	7	100
Score:							

1. Consider the polynomial $P(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$.

[6] (a) Use Descartes' rules of signs to state the number of possible positive and negative roots of $P(x)$

Solution:

$P(x)$ has four sign changes, so the number of possible positive roots is 4, 2, or 0. $P(-x) = (-x)^5 - 2(-x)^4 - 4(-x)^3 + 4(-x)^2 - 5(-x) + 6 = -x^5 - 2x^4 + 4x^3 + 4x^2 + 5x + 6$ has one sign change, so only one negative root is possible.

[4] (b) Use the Rational Root Theorem to list all possible rational roots of $P(x)$.

Solution:

If a rational number $\frac{p}{q}$ is a root of $P(x)$ then q must divide the coefficient of x^5 which equals 1, and p must divide the constant term of $P(x)$ which equals 6. Thus the only possible rational roots are $\pm 1, \pm 2, \pm 3,$ and ± 6.

[12] (c) Given that i is a root of $P(x)$, find the other roots of this polynomial.

Solution:

Since $P(x)$ has real coefficients, $\bar{i} = -i$ is also a root, and it follows that $(x - i)(x + 1) = x^2 + 1$ is a factor of $P(x)$. By using long division, we get that $P(x) = (x^2 + 1)(x^3 - 2x^2 - 5x + 6)$. Since $1^3 - 2(1^3) - 5(1) + 6 = 0$, 1 is a root of $P(x)$. Dividing $x^3 - 2x^2 - 5x + 6$ by $x - 1$, we find that $x^2 - x - 6$ is a factor of $P(x)$. Finally, $x^2 - x - 6 = (x - 3)(x + 2)$, so the roots of $P(x)$ are $i, -i, 1, -2, 3$.

[5] (d) Express $P(x)$ as the product of linear factors only.

Solution:

$P(x) = (x - i)(x + i)(x - 1)(x + 2)(x - 3)$.

2. Consider the vectors $\vec{v_1} = a\vec{\imath} + 2\vec{\jmath} + 4\vec{k}$ and $\vec{v_2} = a\vec{\imath} + 2\vec{\jmath} + a\vec{k}$, where a is a real number.

[7] (a) Determine for which values of a the vectors $\vec{v_1}$ and $\vec{v_2}$ are perpendicular.

Solution:

For $\vec{v_1} \perp \vec{v_2}$ we must have that

$$0 = \vec{v_1} \cdot \vec{v_2} = (a)(a) + (2)(2) + 4a = a^2 + 4a + 4 = (a+2)^2.$$

Therefore, $a = -2$.

[10] (b) Determine the values a for which $\vec{v_1} \times \vec{v_2} = \vec{0}$.

Solution:

$$\vec{v_1} \times \vec{v_2} = \det \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a & 2 & 4 \\ a & 2 & a \end{pmatrix} = \det \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a & 2 & 4 \\ 0 & 0 & a-4 \end{pmatrix}$$

$= (a-4)(-1)^{3+3} \det \begin{pmatrix} \vec{\imath} & \vec{\jmath} \\ a & 2 \end{pmatrix} = (a-4)(2\vec{\imath} - a\vec{\jmath} = (2a-8)\vec{\imath} + (4a-a^2)\vec{\jmath}$. Thus, for $\vec{v_1} \times \vec{v_2} = \vec{0}$ we must have that $2a - 8 = 0$ and $a^2 - 4a = 0$. Obviously, $a = 4$ is the only solution.

3. Let $\mathcal{P}$ denote the plane that is determined by the points $P_1(1,1,0), P_2(1,0,1)$ and $P_3(0,1,1)$. Find

[12] (a) the point-normal equation for $\mathcal{P}$.

Solution:

In order to obtain a vector $\vec{n}$ that is normal to the plane $\mathcal{P}$ we evaluate

$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = [0,-1,1] \times [-1,0,1] = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

$= \vec{i}(-1)^{1+1} \det \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} + (-1)(-1)^{3+1} \det \begin{pmatrix} \vec{j} & \vec{k} \\ 0 & 1 \end{pmatrix}$

$= \vec{i}(-1)(1) - (1)(0) - 1(\vec{j}+\vec{k}) = -(\vec{i}+\vec{j}+\vec{k})$. Therefore, the coordinates of a point $P(x,y,z)$ on the plane must satisfy the equation

$$0 = \overrightarrow{P_1P} \cdot (\vec{i}+\vec{j}+\vec{k}) = (x-1)(1) + (y-1)(1) + (z-0)(1) = x+y+z-2.$$

[5] (b) the equations in symmetric form of the line which is perpendicular to the plane $\mathcal{P}$ and contains the point P_1.

Solution:

The position vector $[x,y,z]$ of a point on the line inquestion is obtained as

$$[x,y,z] = \overrightarrow{OP_1} + s\vec{n} = [1,1,0] + s[1,1,1] = [1+s, 1+s, s].$$

Thus, $x = 1+s, y = 1+s, z = s$ are the scalar parametric equations of the line in question, so $x-1 = y-1 = z$ are the equations in symmetric form.

4. Given the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, indicate whether or not the expressions below are defined. If an expression is defined, evaluate the resulting matrix; if it is not defined, explain why not.

[2] (a) A^T.

Solution:

$$A^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}$$

[3] (b) $A + A^T$.

Solution:

$A + A^T$ is not defined, since A is a $2\times$ 3-matrix and A^T is a 3×2-matrix, so they do not have the same dimensions.

[8] (c) AA^T.

Solution:

$$AA^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1) + (0)(0) + (-1)(-1) & (1)(0) + (0)(-1) + (-1)(1) \\ (0)(1) + (-1)(0) + (1)(-1) & (0)(0) + (-1)(-1) + (1)(1) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

5. Consider the system of linear equations.

$$2x + 6w = 2; \quad z + 7w = 1; \quad 2x + y + 3w = 1; \quad 4x + 15w = 1.$$

[4] (a) Write down the augmented matrix of this system.

Solution:

$$A = \left(\begin{array}{cccc|c} 2 & 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & 7 & 1 \\ 2 & 1 & 0 & 3 & 1 \\ 4 & 0 & 0 & 15 & 1 \end{array}\right).$$

[15] (b) Solve the system by using Gauss-Jordan elimination. Clearly indicate the elementary row operations that you use.

Solution:

$$A \xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 - R_1} \left(\begin{array}{cccc|c} 2 & 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right) \xrightarrow[R_1 \to \frac{1}{2}R_1;\ R_4 \to \frac{1}{3}R_4]{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 3R_4;\ R_2 \to R_2 + 3R_4]{R_3 \to R_3 - 7R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right).$$

Therefore, the solutions are $x = 4, y = -4, z = 8, w = -1$.

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[7] **6.** Evaluate

$$\det\begin{pmatrix} 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 2 & 1 & 0 & 3 \\ 4 & 0 & 0 & 15 \end{pmatrix}.$$

Solution:

$$\det\begin{pmatrix} 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 2 & 1 & 0 & 3 \\ 4 & 0 & 0 & 15 \end{pmatrix} = (1)(-1)^{3+2}\det\begin{pmatrix} 2 & 0 & 6 \\ 0 & 1 & 7 \\ 4 & 0 & 15 \end{pmatrix} = -\det\begin{pmatrix} 2 & 0 & 6 \\ 0 & 1 & 7 \\ 4 & 0 & 15 \end{pmatrix}$$

$$= \ -(1)(-1)^{2+2}\det\begin{pmatrix} 2 & 6 \\ 4 & 15 \end{pmatrix} = -((2)(15)-(6)(4)) = -(30-24) = -6.$$

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