THE UNIVERSITY OF MANITOBA

DATE: December 8, 2012 (6:00 p.m. – 8:00 p.m.) **FINAL EXAMINATION**

DEPARTMENT & COURSE NO: MATH1210 **TIME**: 2 hours

EXAMINATION: Techniques of Classical and Linear Algebra **EXAMINERS**: D. Kalajdzievska

S. Kalajdzievski

D. Trim

INSTRUCTIONS:

1. No aids permitted.

2. Attempt all questions.

3. If insufficient space is provided for a solution to a problem, continue your work on the back of the previous page.

4. Check that your examination booklet contains questions numbered from 1 to 10.

5. Fill in the information requested below.

Student Name (Print): ______________________

Student Signature: ______________________

Student Number: ______________________

Seat Number: ______________________

Circle your instructor's name: D. Kalajdzievska S. Kalajdzievski D. Trim

Question	Maximum Mark	Assigned Mark	Question	Maximum Mark	Assigned Mark
1	6		6	5	
2	5		7	8	
3	6		8	4	
4	5		9	10	
5	3		10	8	
Total	25		Total	35	

Examination Total /60

6 **1.** (a) According to the rational root theorem, what are the possible rational numbers that can satisfy the equation

$$5x^3 + 4x^2 + 4x - 1 = 0?$$

(b) Find all roots of the equation.

(a) Possible rational roots are $\pm 1, \pm\dfrac{1}{5}$.

(b) Since one solution is $x = 1/5$, we can factor $5x - 1$ from the cubic,

$$5x^3 + 4x^2 + 4x - 1 = (5x - 1)(x^2 + x + 1) = 0.$$

The other two solutions are $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$.

5 **2.** Find parametric equations for the line through the point $(2, -1, 3)$ parallel to the line defined by

$$x + y - 2z = 4, \quad 2x - y + 3z = 6.$$

A vector parallel to the line is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix} = \langle 1, -7, -3 \rangle.$$

Parametric equations for the line are therefore

$$x = 2 + t, \quad y = -1 - 7t, \quad z = 3 - 3t.$$

6 **3.** Find all solutions of the system of equations

$$\begin{aligned} x + y + z &= 2, \\ 2x - y + 3z &= -1, \\ 7x + y + 9z &= 4. \end{aligned}$$

The augmented matrix for the system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -1 \\ 7 & 1 & 9 & 4 \end{array}\right) \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -7R_1 + R_3 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & -5 \\ 0 & -6 & 2 & -10 \end{array}\right) R_3 \to -2R_2 + R_3$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) R_2 \to -R_2/3 \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 \to -R_2 + R_1$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4/3 & 1/3 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

When we convert to equations,

$$x + \frac{4z}{3} = \frac{1}{3}, \qquad y - \frac{z}{3} = \frac{5}{3}.$$

Solutions are

$$x = \frac{1}{3} - \frac{4z}{3}, \qquad y = \frac{5}{3} + \frac{z}{3}, \qquad \text{where } z \text{ is arbitrary.}$$

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5 **4.** Use Cramer's rule to find the value of y satisfying the system of equations

$$\begin{aligned} x + 2y - 3z &= 4, \\ 2x + y &= 3, \\ x - y + z &= -1. \end{aligned}$$

$$y = \frac{\begin{vmatrix} 1 & 4 & -3 \\ 2 & 3 & 0 \\ 1 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{-3(-5) + 1(-5)}{-3(-3) + 1(-3)} = \frac{10}{6} = \frac{5}{3}$$

3 **5.** You are given that the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix} \qquad \text{is} \qquad A^{-1} = \frac{1}{8}\begin{pmatrix} 13 & -4 & -1 \\ -15 & 4 & 3 \\ 10 & 0 & -2 \end{pmatrix}.$$

Use the inverse matrix to find the value of z satisfying the system of equations

$$\begin{aligned} x + y + z &= 3, \\ 2y + 3z &= -4, \\ 5x + 5y + z &= 10. \end{aligned}$$

The solution of the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 3 \\ -4 \\ 10 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 13 & -4 & -1 \\ -15 & 4 & 3 \\ 10 & 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \\ 10 \end{pmatrix}.$$

Hence,

$$z = \frac{1}{8}[10(3) - 2(10)] = \frac{10}{8} = \frac{5}{4}.$$

5 **6.** Determine whether the following sets of vectors are linearly dependent or linearly independent. Justify your answers.

(a) $\langle 1, 2, -3\rangle$, $\langle 4, 2, 1\rangle$, $\langle -5, 1, 17\rangle$, $\langle 4, 1, 6\rangle$

(b) $\langle 1, 3, -2\rangle$, $\langle 2, 4, 5\rangle$, $\langle -2, 1, 3\rangle$

(a) Because there are more vectors than components, the vectors must be linearly dependent.

(b) Because the determinant

$$\begin{vmatrix} 1 & 2 & -2 \\ 3 & 4 & 1 \\ -2 & 5 & 3 \end{vmatrix} = 1(7) - 2(11) - 2(23) = -61 \neq 0,$$

the vectors are linearly independent.

8 **7.** The matrix of a linear transformation T is

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}.$$

(a) Find $T\langle 1, -2, 4\rangle$.
(b) Find all eigenvalues of the linear transformation.
(c) Find all eigenvectors associated with the smallest eigenvalue in part (b).

(a) $T\langle 1, -2, 4\rangle = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ 10 \end{pmatrix}$

(b) Eigenvalues are given by the equation

$$\begin{aligned} 0 = |A - \lambda I| &= \begin{vmatrix} 2-\lambda & 2 & 3 \\ 1 & 2-\lambda & 1 \\ 2 & -2 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)[(2-\lambda)(1-\lambda)+2] - 2(1-\lambda-2) + 3[-2-2(2-\lambda)] \\ &= (2-\lambda)(\lambda^2 - 3\lambda + 4) - 2(-\lambda - 1) + 3(-6 + 2\lambda) \\ &= -\lambda^3 + 5\lambda^2 - 2\lambda - 8 = -(\lambda^3 - 5\lambda^2 + 2\lambda + 8) \\ &= -(\lambda+1)(\lambda^2 - 6\lambda + 8) = -(\lambda+1)(\lambda-2)(\lambda-4). \end{aligned}$$

Eigenvalues are $\lambda = -1, 2, 4$. (c) If $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is an eigenvector corresponding to $\lambda = -1$, then

$$\begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_2 \\ R_2 \to R_1 \\ \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 3 & 2 & 3 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \to -3R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & -8 & 0 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \to -R_2/7 \\ R_3 \to -R_3/8 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \to -3R_2 + R_1 \\ \\ R_3 \to -R_2 + R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

These imply that $v_1 = -v_3$ and $v_2 = 0$. Eigenvectors are therefore $\mathbf{v} = \langle -v_3, 0, v_3\rangle = v_3\langle -1, 0, 1\rangle$.

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4 **8.** (a) Show that if λ is an eigenvalue of a matrix A, then λ^2 is an eigenvalue for the matrix A^2.

(b) If $\mathbf{v}$ is an eigenvector corresponding to λ, what is an eigenvector corresponding to λ^2?

If $\mathbf{v}$ is an eigenvector of A, and λ is the corresponding eigenvalue, then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

If we multiply both sides of this equation by A, then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

This shows that λ^2 is an eigenvalue of A^2. It also shows that $\mathbf{v}$ is an eigenvector corresponding to λ^2.

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10 **9.** In the five parts of this question, circle the correct answer.

(a) When the determinant of the coefficient matrix of a system of n linear, homogeneous equations in n unknowns has value zero, the system has

(i) one solution
(ii) no solutions
(iii) an infinity of solutions

(b) The augmented matrix of a system of m equations in n unknowns has r leading ones in its RREF, where $r < n$. The system has

(i) no solutions
(ii) one solution
(iii) an infinity of solutions with $m - r$ parameters
(iv) an infinity of solutions with $n - r$ parameters
(v) an infinity of solutions with an undetermined number of parameters

(c) A system of n equations in m unknowns is known to have exactly one solution. The number of leading ones in the RREF of the augmented matrix for the system is

(i) n
(ii) m
(iii) $n - m$
(iv) $m - n$
(v) none of the above

(d) $(3AB)^T$ is equal to

(i) $3A^T B^T$
(ii) $3B^T A^T$
(iii) $(1/3)A^T B^T$
(iv) $(1/3)B^T A^T$
(v) none of the above

(e) If A is an invertible matrix, then

(i) $(A^2)^{-1} = (A^{-1})^2$
(ii) $(A^2)^{-1} = \dfrac{1}{A^2}$
(iii) A^2 does not have an inverse
(iv) A^2 has an inverse but it is not (i) or (ii)

Answers are : (a) (iii) (b) (iv) (c) (ii) (d) (ii) (e) (i)

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8 **10.** Use mathematical induction to prove that 576 divides $5^{2n+2} - 24n - 25$ for $n \geq 1$.

When $n = 1$, $5^{2n+2} - 24n - 25 = 5^4 - 24 - 25 = 576$, and this is clearly divisible by 576. The result is therefore true for $n = 1$. Suppose the result is valid for some integer k; that is, suppose that $5^{2k+2} - 24k - 25$ is divisible by 576. We must now show that $5^{2k+4} - 24(k+1) - 25$ is divisible by 576. Now,

$$\begin{aligned}
5^{2k+4} - 24(k+1) - 25 &= 25 \cdot 5^{2k+2} - 24k - 49 \\
&= 25(5^{2k+2} - 24k - 25) - 24k - 49 + 25(24k + 25) \\
&= 25(5^{2k+2} - 24k - 25) + 576k - 576.
\end{aligned}$$

Since 576 divides each term on the right side of this equation, it follows that 576 divides $5^{2k+4} - 24(k+1) - 25$. The result is therefore valid for $k + 1$, and hence, by the principle of mathematical induction, the result is valid for all $n \geq 1$.