Midterm Examination for Math 1210 **60 minutes**

INSTRUCTIONS:

1. No aids permitted.

2. Attempt all questions.

3. If insufficient space is provided for a solution to a problem, continue your work on the blank page at end of the examination. Clearly indicate that you have done so. Do not remove this page.

4. Check that your examination booklet contains ten questions.

5. Fill in the information requested below.

Student Name (Print): ______________________________

Student Signature: ______________________________

Student Number: ______________________________

Circle your instructor's name: D. Kalajdzievska S. Kalajdzievski D. Trim

Question	Maximum Mark	Assigned Mark	Question	Maximum Mark	Assigned Mark
1	10		6	4	
2	6		7	4	
3	5		8	4	
4	6		9	3	
5	3		10	3	
Total	30		Total	18	

Examination Total /48

10 **1.** Use mathematical induction to prove that

$$1 + 2(2) + 3(2^2) + 4(2^3) + \cdots + n(2^{n-1}) = 1 + (n+1)2^n - 2^{n+1}, \quad n \geq 1.$$

When $n = 1$, L.S. $= 1$ and the R.S. $= 1 + (2)2^1 - 2^2 = 1$. Thus, the formula is correct for $n = 1$. Suppose that the formula is valid for some integer k; that is,

$$1 + 2(2) + 3(2^2) + 4(2^3) + \cdots + k(2^{k-1}) = 1 + (k+1)2^k - 2^{k+1}.$$

We must now prove that the formula is valid for $k+1$, that is, we must prove that

$$1 + 2(2) + 3(2^2) + 4(2^3) + \cdots + (k+1)(2^k) = 1 + (k+2)2^{k+1} - 2^{k+2}.$$

The left side of this equation is

$$\begin{aligned}
\text{L.S.} &= [1 + 2(2) + 3(2^2) + 4(2^3) + \cdots + k(2^{k-1}) + (k+1)(2^k) \\
&= [1 + (k+1)2^k - 2^{k+1}] + (k+1)(2^k) \\
&= 1 + 2(k+1)2^k - 2^{k+1} \\
&= 1 + (k+1)2^{k+1} - 2^{k+1} \\
&= 1 + (k+2)2^{k+1} - 2^{k+1} - 2^{k+1} \\
&= 1 + (k+2)2^{k+1} - 2^{k+2}.
\end{aligned}$$

Since the formula has been proved for $k+1$, it follows by the principle of mathematical induction, that is valid for all $n \geq 1$.

6 **2.** Evaluate the summation

$$\sum_{i=20}^{45} (4i^2 - 1).$$

Do not simplify your answer. You may use any of the following formulas,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

We write that

$$\begin{aligned}\sum_{i=20}^{45} (4i^2 - 1) &= \sum_{i=1}^{45} (4i^2 - 1) - \sum_{i=1}^{19} (4i^2 - 1) \\ &= \left[4\sum_{i=1}^{45} i^2 - 45\right] - \left[4\sum_{i=1}^{19} i^2 - 19\right] \\ &= \left[4\frac{45(46)(91)}{6} - 45\right] - \left[4\frac{19(20)(39)}{6} - 19\right].\end{aligned}$$

5 **3.** Find the imaginary part of the complex number

$$(1 - \sqrt{3}\, i)^{10},$$

simplified as much as possible.

Since $1 - \sqrt{3}\, i = 2e^{-\pi i/3}$,

$$(1 - \sqrt{3}\, i)^{10} = [2e^{-\pi i/3}]^{10} = 2^{10} e^{-10\pi i/3}.$$

The imaginary part of this complex number is

$$2^{10} \sin\left(\frac{-10\pi}{3}\right) = 2^{10} \sin\left(\frac{2\pi}{3}\right) = 2^{10}\left(\frac{\sqrt{3}}{2}\right) = 2^9\sqrt{3}.$$

6 **4.** Find all solutions of the equation

$$x^4 + 1 = 0.$$

Express any complex solutions in Cartesian form, simplified as much as possible.

We write

$$x^4 = -1 = 1e^{\pi i} = e^{\pi i + 2k\pi i} = e^{(2k+1)\pi i}.$$

When we take fourth roots, we obtain

$$x = [e^{(2k+1)\pi i}]^{1/4} = e^{(2k+1)\pi i/4}.$$

For $k = 0, 1, 2, 3$, we obtain

$$\begin{aligned}
x_0 &= e^{\pi i/4} = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) i = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \\
x_1 &= e^{\pi 3i/4} = \cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) i = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \\
x_2 &= e^{\pi 5i/4} = \cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right) i = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \\
x_3 &= e^{\pi 7i/4} = \cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right) i = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.
\end{aligned}$$

3 **5.** If A and B are the matrices

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 6 & 7 \\ 2 & 1 & 5 \end{pmatrix},$$

what is the $(2, 3)$ entry of $A^2 + 2B^T$; that is, the entry in the second row and third column.

If we write

$$A^2 + 2B^T = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 5 \end{pmatrix} + 2 \begin{pmatrix} 1 & 4 & 2 \\ -2 & 6 & 1 \\ 3 & 7 & 5 \end{pmatrix},$$

we see that the $(2, 3)$ entry is

$$[-1(0) + 4(2) + 2(5)] + 2(1) = 20.$$

4 **6.** Find a vector of length 3 that is perpendicular to both the y-axis and the vector $\langle 3, -1, 2\rangle$.

Since a vector along the y-axis is $\hat{\mathbf{j}} = \langle 0, 1, 0\rangle$, a vector perpendicular to the y-axis and $\langle 3, -1, 2\rangle$ is

$$\hat{\mathbf{j}} \times \langle 3, -1, 2\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 3 & -1 & 2 \end{vmatrix} = \langle 2, 0, -3\rangle.$$

A unit vector in this direction is

$$\frac{\langle 2, 0, -3\rangle}{\sqrt{(2)^2 + (-3)^2}} = \frac{1}{\sqrt{13}}\langle 2, 0, -3\rangle.$$

A vector of length 3 in this direction is

$$\frac{3}{\sqrt{13}}\langle 2, 0, -3\rangle.$$

4 **7.** Find the equation of the plane that passes through the point $(-1, 2, 4)$ and is parallel to the plane $8x - 4y + 2z = 11$. Simplify the equation as much as possible.

A vector normal to the given plane is $\langle 8, -4, 2\rangle$. Since the required plane is parallel to this plane, this must also be a normal vector to the required plane. The equation of the required plane is therefore

$$\begin{aligned} 8(x+1) - 4(y-2) + 2(z-4) &= 0 \\ 8x - 4y + 2z &= -8 \\ 4x - 2y + z &= -4 \end{aligned}$$

4 **8.** What do Descartes' rules of sign predict for the number of positive and negative roots of the equation

$$2x^5 - 3x^3 - 5x = -1?$$

Since $P(x) = 2x^5 - 3x^3 - 5x + 1$ has 2 sign changes, there is either 2 positive roots or no positive roots. Since $P(-x) = -2x^5 + 3x^3 + 5x + 1$ has one sign change, there is exactly 1 negative root.

3 **9.** If x is a complex number that satisfies the equation

$$3x^5 + 2x^4 - 10x^2 + 5 = 0,$$

what does the bounds theorem predict about the modulus of x?

Since $M = 10$, the bounds theorem says that

$$|x| < \frac{10}{3} + 1 = \frac{13}{3}.$$

3 **10.** According to the rational root theorem, what are the possible rational numbers that can satisfy the equation

$$4x^3 - 4x^2 + 5x + 10 = 0?$$

Since divisors of 10 are $\pm 1, \pm 2, \pm 5, \pm 10$, and divisors of 4 are $\pm 1, \pm 2, \pm 4$, possible rational roots are

$$\pm 1,\ \pm 2,\ \pm 5,\ \pm 10,\ \pm \frac{1}{2},\ \pm \frac{5}{2},\ \pm \frac{1}{4},\ \pm \frac{5}{4}.$$