**UNIVERSITY OF MANITOBA**

DATE: December 11, 2013 — FINAL EXAMINATION

TITLE PAGE

EXAMINATION: Techniques of Classical and Linear Algebra — TIME: 2 hours

COURSE: MATH 1210 — EXAMINER: Davidson, Harland, Moghaddam

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FAMILY NAME: (Print in ink) ______________________

GIVEN NAME: (Print in ink) ______________________

STUDENT NUMBER: __________

SEAT NUMBER: __________

SIGNATURE: (in ink) ______________________

(I understand that cheating is a serious offense)

- ❑ A01 9:30–10:20 AM MWF (200 Armes) M. Davidson
- ❑ A02 1:30–2:20 PM MWF (204 Armes) G. I. Moghaddam
- ❑ A03 1:30–2:20 PM MWF (100 St. Paul) N. Harland

**INSTRUCTIONS TO STUDENTS:**

This is a 2 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 11 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 90 points.

**Answer questions 1 to 9**. Question 10 is a bonus question and you have choice to answer or ignore.
**Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

| Question | Points | Score |
|---|---|---|
| 1 | 18 | |
| 2 | 7 | |
| 3 | 7 | |
| 4 | 8 | |
| 5 | 10 | |
| 6 | 9 | |
| 7 | 10 | |
| 8 | 9 | |
| 9 | 12 | |
| Total: | 90 | |

1. The following are short answer questions.

[3] (a) Let $\overrightarrow{u} = \langle -2, 5, -2\rangle$ and $\overrightarrow{v} = \overrightarrow{AB}$ where $A(3, 7, 1)$ and $B(4, 6, 4)$ are two points. Find the value of $\cos\theta$ where $\theta$ is the angle between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$.

[3] (b) Use Cramer's rule to find the value of $y$ **only** for the linear system

$$\begin{aligned}\frac{1}{2}x + y &= 7\\ 6x - 2y &= 18\,.\end{aligned}$$

(Do not use any other method)

[2] (c) Suppose the coefficient matrix of a homogeneous system is a $6 \times 9$ matrix of rank 4. How many linearly independent basic solutions will it have?

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[2] (d) Are the vectors $\{\langle 1,3,7\rangle, \langle 3,2,-5\rangle, \langle 4,-1,6\rangle, \langle 2,-7,2\rangle\}$ linearly dependent or linearly independent? Justify your answer.

[3] (e) Use formula $\displaystyle\sum_{k=1}^{m} k^2 = \frac{1}{6}\left[m(m+1)(2m+1)\right]$ to evaluate the sum $\displaystyle\sum_{j=2}^{11}\left[j^2-2j+1\right]$.

[2] (f) Is the transformation $T: \begin{aligned} v_1' &= -2v_1 - v_2 + v_3 \\ v_2' &= v_1 - 4v_2 \\ v_3' &= v_1 - v_2 + v_3 \end{aligned}$ a linear transformation? Why?

[3] (g) Let $T: \begin{aligned} v_1' &= v_1 - v_2 \\ v_2' &= v_1 + v_2 \end{aligned}$ be a linear transformation. Is it true that $T\langle 2,5\rangle = 2\,T\langle 1,0\rangle + 5\,T\langle 0,1\rangle$ ? Why?

[7] 2. Let $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$; use mathematical induction on positive integer $n \geq 1$ to prove that $A^n = \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix}$.

[7] 3. Find all real values of $x$ such that

$$\frac{3+i}{1-i} - i^{2012} = \overline{-\sqrt{5}-2i} + \Big| 1+xi \Big|.$$

4. Let $P(x) = 2x^3 + 3x^2 + 5x + 4$.

[3] (a) Show that $P(x)$ has no real root in the interval $[1, 3]$.

[5] (b) Find all roots of the polynomial $P(x)$.

5. Let $A = \begin{pmatrix} 3 & -2 & -1 \\ 2 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -2 \\ 1 & 2 \\ -1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$,

$E = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ and $F = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$.

[6] (a) Indicate if the expression is defined or undefined by placing a check mark (✓) in the appropriate column. If it is defined, then indicate its size.

| EXPRESSION | UNDEFINED | DEFINED | SIZE |
|---|---|---|---|
| $B^T A^T + 2C$ | | | |
| $EA + F^T$ | | | |
| $B^T B + D$ | | | |
| $AF + (EB)^T$ | | | |

[4] (b) Evaluate $AD + B^T$

6. Consider the vectors:

$$\overrightarrow{u_1} = \langle 1, -1, 0 \rangle \qquad \overrightarrow{u_2} = \langle 0, 1, 1 \rangle \qquad \overrightarrow{u_3} = \langle 2, -2, 1 \rangle$$

[3] (a) Show that the vectors $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ are linearly independent. (Justify your answer.)

[6] (b) Write $\overrightarrow{v} = \langle -4, 11, 4 \rangle$ as a linear combination of $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$.

7. For the matrix $A = \begin{pmatrix} 2 & -4 & 1 \\ 0 & 2 & 4 \\ 1 & -2 & 0 \end{pmatrix}$ :

[7] (a) Use row reduction to find $A^{-1}$ if it exists.

[3] (b) Use the information from part $(a)$ to solve the system

$$\begin{aligned} 2x - 4y + z &= 1 \\ 2y + 4z &= -1 \\ x - 2y &= 2. \end{aligned}$$

(Do not use any other method.)

8. The matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -2 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$

has determinant of $-21$ (you do not need to show this). The adjoint is

$$adj(A) = \begin{pmatrix} -1 & a & 4 \\ -8 & 2 & 11 \\ b & 3 & 6 \end{pmatrix}.$$

[6] (a) Find the values of $a$ and $b$.

[3] (b) Use all the above to find $A^{-1}$.

9. Let $A$ be the matrix $A = \begin{pmatrix} 1 & -12 & 3 \\ 0 & -3 & 1 \\ 0 & -4 & 2 \end{pmatrix}$.

[6] (a) Find all eigenvalues of $A$.

[6] (b) Find the eigenvectors corresponding to $\lambda = 1$ .

**This is a bonus question. You do not have to answer it**.

[5] 10. Show that if $\lambda$ is an eigenvalue of a matrix $A$ which satisfies $A^3 = A$, then $\lambda = 0,\ 1$ or $-1$.

BLANK PAGE FOR ROUGH WORK