UNIVERSITY OF MANITOBA

DATE: October 24, 2013

MIDTERM
TITLE PAGE
TIME: 1 hour

EXAMINATION: Techniques of Classical and Linear Algebra
COURSE: MATH 1210 EXAMINER: Davidson, Harland, Moghaddam

NAME: (Print in ink) ANSWER KEY
STUDENT NUMBER: ______
SIGNATURE: (in ink) ______
(I understand that cheating is a serious offense)

- ☐ A01 9:30–10:20 AM MWF (200 Armes) M. Davidson
- ☐ A02 1:30–2:20 PM MWF (204 Armes) G. I. Moghaddam
- ☐ A03 1:30–2:20 PM MWF (100 St. Paul) N. Harland

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 7 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 55 points.

**Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

| Question | Points | Score |
|---|---|---|
| 1 | 11 | |
| 2 | 6 | |
| 3 | 16 | |
| 4 | 7 | |
| 5 | 8 | |
| 6 | 7 | |
| Total: | 55 | |

[11] 1. (a) Use mathematical induction on integer $n \geq 1$ to prove that

$$2+5+8+\ldots+(6n-1)=n(6n+1)\,.$$

Let $P_n : 2+5+8+\cdots+(6n-1) = n(6n+1)$

If $n=1$ then $6(1)-1=5$

Now $2+5=7$ and $(1)(6(1)+1)=7$

Hence $P_1$ is true.

Suppose $P_k$ is true, ie. $2+5+8+\cdots+(6k-1)=k(6k+1)$

(We want to show $P_{k+1}$ is also true:

$2+5+8+\cdots+(6(k+1)-1)=(k+1)(6(k+1)+1)$ )

Now $2+5+8+\cdots+(6(k+1)-1)$

$= 2+5+8+\cdots+(6k+5)$

$= 2+5+8+\cdots+(6k-1)+(6k+2)+(6k+5)$

$= k(6k+1)+12k+7$

$= 6k^2+13k+7$

$= (k+1)(6k+7) = (k+1)(6(k+1)+1)$

Hence $P_{k+1}$ is also true.

Since $P_1$ is true and $P_k$ being true implies $P_{k+1}$ is also true, by PMI, $P_n$ is true for all $n \geq 1$.

(b) Write $2+5+8+\ldots+(6n-1)$ in sigma notation.

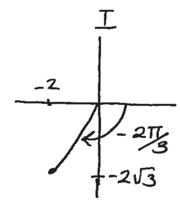
$$\sum_{i=1}^{2n} (3i-1)$$

[6] 2. Find all fourth roots of $-2-2\sqrt{3}\,i$. Leave your answer in exponential form.

We want all solutions to

$$z^4 = -2-2\sqrt{3}\,i$$

$$|-2-2\sqrt{3}i| = \sqrt{(-2)^2+(-2\sqrt{3})^2}$$
$$= \sqrt{4+12} = \sqrt{16} = 4$$




$$\arg(-2-2\sqrt{3}i) = -\frac{2\pi}{3} \quad \left(\text{or } \frac{4\pi}{3}\right)$$

If $z = re^{\theta i}$ then $z^4 = r^4 e^{4\theta i}$

So $$r^4 e^{4\theta i} = 4e^{-\frac{2\pi}{3}i}$$

$r^4 = 4$

$r = \sqrt{2}$

$$4\theta = -\frac{2\pi}{3} + 2n\pi$$

$$\theta = -\frac{2\pi}{12} + \frac{2n\pi}{4}$$
$$= \frac{-2\pi + 6n\pi}{12}$$
$$= \frac{-\pi + 3n\pi}{6}$$

$\theta_0 = -\pi/6$

$\theta_1 = 2\pi/6 = \pi/3$

$\theta_2 = 5\pi/6$

$\theta_3 = 8\pi/6 = \frac{4\pi}{3}$

(in principle value $\theta_3 = -\frac{2\pi}{3}$)

The four roots are:

$$\sqrt{2}\,e^{-\pi/6\, i},\ \sqrt{2}\,e^{\pi/3\, i},\ \sqrt{2}\,e^{5\pi/6\, i},\ \sqrt{2}\,e^{-2\pi/3\, i}$$

[16] 3. Consider the polynomial equation of $P(x) = 0$ where

$$P(x) = 3x^4 - 8x^3 + 4x^2 + 25$$

(a) What are the possible rational zeros of $P(x)$?

The possible rational roots are

$$\left\{ \pm 1, \pm 5, \pm 25, \pm \tfrac{1}{3}, \pm \tfrac{5}{3}, \pm \tfrac{25}{3} \right\}$$

(b) Use Descartes' rule of signs to find the possible number of positive and negative roots of $P(x)$.

$P(x)$ has 2 sign changes, so $P(x)$ has 2 or 0 positive real roots.

$$P(-x) = 3x^4 + 8x^3 + 4x^2 + 25$$

$P(-x)$ has 0 sign changes, so $P(x)$ has no negative real roots.

(c) Use Bounds Theorem to find a bound on the roots of $P(x)$.

If $P(\alpha) = 0$ then

$$|\alpha| < \frac{25}{3} + 1 = \frac{28}{3} \left(= 9 \tfrac{1}{3}\right)$$

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[*Recall that* $P(x) = 3x^4 - 8x^3 + 4x^2 + 25$]

(d) Update the list from part (a) using the information from parts (b) and (c).

Possible Rational Root Remaining:

$\{1, 5, 1/3, 5/3, 25/3\}$

(e) Given that $2 + i$ is a root of $P(x)$, find all the roots of $P(x)$.

Since $2+i$ is a root, and $P(x)$ is a real polynomial, $2-i$ is also a root.

Hence $(x-(2+i))(x-(2-i)) = x^2-4x+5$ is a factor.

$$\begin{array}{r|l} & 3x^2 + 4x + 5 \\ x^2-4x+5 & 3x^4 - 8x^3 + 4x^2 + 25 \\ & \underline{3x^4 - 12x^3 + 15x^2} \\ & 4x^3 - 11x^2 \\ & \underline{4x^3 - 16x^2 + 20x} \\ & 5x^2 - 20x + 25 \\ & \underline{5x^2 - 20x + 25} \\ & 0 \end{array}$$

So $P(x) = (x^2-4x+5)(3x^2+4x+5)$

$3x^2+4x+5$ has roots $x = \dfrac{-4 \pm \sqrt{16-4(3)(5)}}{2(3)} = \dfrac{-4 \pm \sqrt{16-60}}{6}$

$= \dfrac{-4 \pm \sqrt{-44}}{6} = \dfrac{-2 \pm \sqrt{11}\, i}{3}$

Hence the roots of $P(x)$ are:

$2+i,\ 2-i,\ -\frac{2}{3} + \frac{\sqrt{11}}{3}i,\ -\frac{2}{3} - \frac{\sqrt{11}}{3}i$

[7] 4. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$. Evaluate each of the following expressions or explain why it is not defined.

(a) $B(A - A^T)$.

Since A is a 2×7 matrix, $A^T$ is a 7×2 matrix. Matrix subtraction is only defined if the matrices are the same size, so this expression is undefined.

(b) $(B + B^T)A$.

$$\left(\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\right)\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 & 12 & 16 & 20 & 24 & 28 \\ -6 & 0 & 6 & 0 & -6 & 0 & 6 \end{pmatrix}$$

[8] 5. Let $\mathbf{u} = \langle a, b, c \rangle$, $\mathbf{v} = \langle 1, 2, -1 \rangle$ and $\mathbf{w} = \langle 3, 1, 5 \rangle$.

(a) Find all values of $a$ for which $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = b^2 + c^2$.

$$(u+v)\cdot(u-v) = b^2+c^2$$

$$\langle a+1, b+2, c-1\rangle \cdot \langle a-1, b-2, c+1\rangle = b^2+c^2$$

$$(a+1)(a-1) + (b+2)(b-2) + (c-1)(c+1) = b^2+c^2$$

$$a^2-1+b^2-4+c^2-1 = b^2+c^2$$

$$a^2 = 6$$

$$a = \pm\sqrt{6}$$

The values of $a$ which satisfy the above are $\sqrt{6}$ and $-\sqrt{6}$.

(b) Find the angle between $-2\mathbf{v}$ and $3\mathbf{w}$.

$-2\vec{v} = \langle -2, -4, 2\rangle$ and $3\vec{\omega} = \langle 9, 3, 15\rangle$

$(-2\vec{v})\cdot(3\vec{\omega}) = -18-12+30 = 0$; hence the angle between the vectors is $\pi/2$.

Alternately: The angle between $-2\vec{v}$ and $3\vec{\omega}$ is the same as the angle between $-\vec{v}$ and $\vec{\omega}$;

$\langle -1, -2, 1\rangle \cdot \langle 3, 1, 5\rangle = -3-2+5 = 0$,

Hence the angle is $\pi/2$

[7] 6. Consider the point $P(4,1,-3)$ and the two lines $L_1 : x = t,\ y = 1+2t,\ z = 1$ and $L_2 : x = -r,\ y = 6+3r,\ z = 2+r$.

(a) Find parametric equations of the line through the point $P$ and perpendicular to both lines $L_1$ and $L_2$.

A vector in the direction of $L_1$ is $\langle 1,2,0\rangle = \vec{V_1}$
A vector in the direction of $L_2$ is $\langle -1,3,1\rangle = \vec{V_2}$
$\vec{V_1} \times \vec{V_2}$ is perpendicular to both $\vec{V_1}$ and $\vec{V_2}$, so

$$\vec{V_1} \times \vec{V_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ -1 & 3 & 1 \end{vmatrix} = \langle 2,-1,5\rangle$$

So the desired parametric equations are:

$$x = 4+2t$$
$$y = 1-t$$
$$z = -3+5t$$

(b) Find an equation of the plane through the point $P$ and parallel to both lines $L_1$ and $L_2$.

A plane that is parallel to $L_1$ will have its normal vector perpendicular to $\vec{V_1}$, Similarly with $\vec{V_2}$; hence the normal to this plane is $\langle 2,-1,5\rangle$

So the plane is

$$\langle 2,-1,5\rangle \cdot \langle x-4, y-1, z+3\rangle = 0$$

or $2(x-4) - (y-1) + 5(z+3) = 0$

or $2x - y + 5z = -8$