UNIVERSITY OF MANITOBA

DATE: October 23, 2014
COURSE: MATH 1210
EXAMINATION:
Techniques of Classical and Linear Algebra

MIDTERM EXAMINATION
TITLE PAGE
TIME: 60 Minutes
EXAMINER: Various

NAME: (Print in ink) ______________________

STUDENT NUMBER: __________

SIGNATURE: (in ink) ______________________

(I understand that cheating is a serious offense.)

- ❑ A01 9:30-10:20 MWF J. Arino
- ❑ A02 13:30–14:20 MWF X. Zhao
- ❑ A03 13:30–14:20 MWF J. Arino

INSTRUCTIONS TO STUDENTS:

This is a 60 Minute exam. **Please show your work clearly.**

No texts, notes, calculators, cellphones, translators or any other electronic devices are permitted.

This exam has a title page, 5 pages of questions (including 1 blank page for rough work). Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staples.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, or on a page at the back, but CLEARLY INDICATE that your work is continued and label the continuation.

Question	Points	Score
1	11	
2	12	
3	4	
4	10	
5	8	
6	5	
Total:	50	

[8] 1. (a) Use mathematical induction to prove that

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots+2n(2n+1)=\frac{4n(n+1)(2n+1)}{3},\ \text{for all } n\geq 1.$$

Solution: Denote the given statement by P_n. [1 mark]

(Base step) When $n=1$, $1\cdot 2+2\cdot 3=8$ and $4(1+1)(2+1)/3=8$. So P_1 is valid. [1 mark]

(Inductive step) Suppose P_k is valid for some $k\geq 1$, i.e.,

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots+2k(2k+1)=\frac{4k(k+1)(2k+1)}{3}. \qquad \text{[2 marks]}$$

We want to prove that P_{k+1} is also valid, i.e.,

$$1\cdot 2+2\cdot 3+\cdots+2(k+1)(2(k+1)+1)=\frac{4(k+1)(k+1+1)(2(k+1)+1)}{3}$$

LHS of P_{k+1} is:

$$\begin{aligned}
&1\cdot 2+2\cdot 3+3\cdot 4+\cdots+2k(2k+1)+(2k+1)(2k+2)+(2k+2)(2k+3)\\
=&\frac{4k(k+1)(2k+1)}{3}+(2k+1)(2k+2)+(2k+2)(2k+3)\\
=&\frac{4k(k+1)(2k+1)+6(2k+1)(k+1)+6(k+1)(2k+3)}{3}\\
=&\frac{(k+1)(4k(2k+1)+6(2k+1)+6(2k+3))}{3}\\
=&\frac{(k+1)(8k^2+28k+24)}{3}\\
=&\frac{4(k+1)(k+2)(2k+3)}{3}
\end{aligned}$$

RHS of P_{k+1}:

$$\frac{4(k+1)(k+1+1)(2(k+1)+1)}{3}=\frac{4(k+1)(k+2)(2k+3)}{3}$$

Thus P_{k+1} is valid. [3 marks; had to include the previous sentence for full marks]

Therefore, by the principle of mathematical induction, P_n is valid for all $n\geq 1$. [1 mark]

Remark: marks for a correct statement are indicated. It was easy to obtain 4 marks just by following the script. With the checking of P_1 being valid, that means 5 marks could be obtained before anything complicated was undertaken.

[3] (b) Write $2+6+12+\cdots+2n(2n+1)$ in sigma notation.

Solution: $\displaystyle\sum_{k=1}^{2n} k(k+1)$.

Beware: $\sum_{k=1}^{2n} k(k+1) \neq \sum_{k=1}^{2n} n(n+1)$, so be careful what letters you use where..

2. Let $f(x) = 6x^4 + kx^3 + 18x^2 + 17x + 4$, where k is an unknown real number. When $f(x)$ is divided by $2x+1$, the remainder is -1.

[3] (a) Find the value of k.

Solution: By Remainder Theorem, $f(-1/2) = -1$, that is,

$$6\left(-\frac{1}{2}\right)^4 + k\left(-\frac{1}{2}\right)^3 + 18\left(-\frac{1}{2}\right)^2 + 17\left(-\frac{1}{2}\right) + 4 = -1.$$

Solving it for k gives $k = 11$.

Remark: we can also use long division to find k. However, the computation is much more complicated.

[3] (b) Use Descartes' rule of signs to find the possible numbers of positive and negative zeros of $f(x)$.

Solution: Since the coefficients of $f(x)$ have no sign changes, $f(x)$ has no positive zeros.

Since the coefficients of $f(-x) = 6x^4 - 11x^3 + 18x^2 - 17x + 4$ have 4 sign changes, $f(x)$ has 4, 2 or 0 negative zeros.

[3] (c) Use bounds theorem to find bounds for zeros of $f(x)$.

Solution: Since $M = \max\{11, 18, 17, 4\} = 18$, all zeros x of $f(x)$ satisfy $|x| < \frac{18}{6} + 1 = 4$.

[3] (d) Taking the results of (b), (c) and the given condition into account, use the rational root theorem to list all possible rational zeros of f.

Solution: The rational root theorem gives possible rational roots:

$$\pm 1, \pm 2, \pm 4, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{1}{6}.$$

Results of (b) reject all positive rationals and results of (c) reject -4. The given condition "when $f(x)$ is divided by $2x+1$ the remainder is -1" implies $-\frac{1}{2}$ is not a root (Remainder Theorem).

Thus, all possible rational zeros are: $-1, -2, -\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, -\frac{1}{6}$

[4] 3. Write the complex expression in Cartesian form: $\dfrac{1}{4i}\left(\dfrac{1+i}{\sqrt{2}}\right)^{48}$.

Solution: $\left|\dfrac{1+i}{\sqrt{2}}\right| = 1$ and $\arg\left(\dfrac{1+i}{\sqrt{2}}\right) = \pi/4$. Hence

$$\frac{1}{4i}\left(\frac{1+i}{\sqrt{2}}\right)^{48} = \frac{1}{4i}\left(e^{\frac{\pi}{4}i}\right)^{48} = \frac{1}{4i}\left(e^{12\pi i}\right) = \frac{1}{4i} = -\frac{1}{4}i.$$

4. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 2 & 1 & 4 & 1 & 0 \\ -1 & 3 & 4 & 1 & 6 \end{pmatrix}.$$

Evaluate each of the following expressions or explain why it is undefined:

[3] (a) The (2,3) cofactor of A.

Solution:

$$c_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = -3$$

[3] (b) The 3rd row of $B^T A$.

Solution: Note that the 3rd row of $B^T A$ is the multiplication of the 3rd row of B^T and A, i.e.,

$$\begin{pmatrix} 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 16 & 16 & 4 \end{pmatrix}.$$

[4] (c) $\det(2(A^2)^T)$.

Solution: First calculate $|A|$ by expansion along the first row:

$$|A| = 1 \cdot (-1-6) - 0 + 1 \cdot (12 - 0) = 5.$$

Then

$$\left|2(A^2)^T\right| = 2^3 \cdot \left|(A^2)^T\right| = 2^3 \cdot |A^2| = 2^3 \cdot |A|^2 = 2^3 \cdot 5^2 = 200.$$

Remark: if A is an $n \times n$ matrix and c is a constant, then $|cA| = c^n|A|$.

[8] 5. Using Cramer's rule, solve the linear system

$$\begin{array}{rrrl} 2x_1 & -3x_2 & +x_3 & = 1 \\ & x_2 & -3x_3 & = 2 \\ & & 2x_3 & = 0. \end{array}$$

No marks will be given for any other method.

Solution: Let A be the coefficient matrix and A_i be the matrix obtained from A by replacing its ith column by $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $i = 1, 2, 3$.

$$|A| = \begin{vmatrix} 2 & -3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 2 \cdot 1 \cdot 2 = 4, \quad |A_1| = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 2(1+6) = 14$$

$$|A_2| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 2 \cdot 2 \cdot 2 = 8, \quad |A_3| = \begin{vmatrix} 2 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Thus $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{14}{4} = \dfrac{7}{2}$, $x_2 = \dfrac{|A_2|}{|A|} = \dfrac{8}{4} = 2$, $x_3 = \dfrac{|A_3|}{|A|} = \dfrac{0}{4} = 0$.

[5] 6. Given that

$$A = \begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix},$$

find **all** values of k for which $A^2 - 7A + 10I_2 = 0_{22}$, where I_2 is the 2×2 identity matrix and 0_{22} is the 2×2 zero matrix.

Solution: Direct substitution gives

$$\begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix}^2 - 7\begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 25 & 10+2k \\ 0 & k^2 \end{bmatrix} - \begin{bmatrix} 35 & 14 \\ 0 & 7k \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2k-4 \\ 0 & k^2-7k+10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $2k - 4 = 0$ and $k^2 - 7k + 10 = 0$. Hence $k = 2$.

Remark: One common error is factoring the given equation as $(A-2)(A-5) = 0$. We cannot do this because $A - 2$ and $A - 5$ are undefined.

Instead, we can factor it as $(A - 2I)(A - 5I) = 0$. However, this factorization is not helpful, because $(A - 2I)(A - 5I) = 0$ does NOT imply $A - 2I = 0$ or $A - 5I = 0$. More generally, $BC = 0$ does not imply $B = 0$ or $C = 0$.

For example, if $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $BC = 0_{22}$, but none of B and C is a zero matrix.