UNIVERSITY OF MANITOBA

DATE: December 21, 2015

FINAL EXAMINATION

TITLE PAGE

DEPARTMENT & COURSE NO: MATH 1210

TIME: 3 hours

EXAMINATION: Techniques for Classical and Linear Algebra

EXAMINER: Borgersen

Unique Identifier Sticker:

DO NOT WRITE ABOVE THIS LINE.

INSTRUCTIONS TO STUDENTS:

This is a 3 hour exam. **Please show your work clearly.**

No texts or notes are permitted. No calculators are permitted. Cell phones, electronic translators, and other electronic devices are **not** permitted.

This exam has a title page and 14 pages of questions, including 2 blank pages for rough/extra work. Please check that you have all the pages.

The value of each question is indicated beside the statement of the question. The total value of all questions is 111 points.

If you need more scrap paper, use the back of the question pages. **Anything written on the back of a page will not be marked.** If you need more space to answer a question (that you want marked), write it on one of the scrap pages at the back.

INDICATE YOUR SECTION:

- ❑ A01 (9:30-10:20 MWF) R. Borgersen
- ❑ A02 (13:30–14:20 MWF) J. Arino
- ❑ A03 (13:30–14:20 MWF) R. Borgersen

FIRST NAME: ____________________

LAST NAME: ____________________

STUDENT NUMBER: ______________

PAPER NUMBER: ______________
(Paper Number is found at the top)

SIGNATURE: (in ink)

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(I understand that cheating is a serious offense)

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1. [12 points] Let $c \in \mathbb{R}$. Prove by induction that for all $n \geq 1$,

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & cn \\ 0 & 1 \end{bmatrix}.$$

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2. Write the following sums using sigma notation with indexes starting at 1 (do not evaluate):

   (a) [3 points] $0.9 + 0.99 + 0.999 + \cdots + 0.999999999$

   (b) [4 points] $\displaystyle\sum_{j=-2}^{6} \frac{2^{j+6}}{\sqrt{j+4}}$

   (c) [3 points] $\dfrac{2(3)}{1(4)} + \dfrac{6(7)}{5(8)} + \dfrac{10(11)}{9(12)} + \dfrac{14(15)}{13(16)} + \cdots + \dfrac{414(415)}{413(416)}$

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3. Write the following complex numbers in Cartesian form:

   (a) [5 points] $\dfrac{\frac{i}{3+i}}{(2-i)^2}$.

   (b) [5 points] $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^{75}$ (Hint: Use DeMoivre's theorem)

4. [8 points] Let $f(x) = 2x^4 - x^3 + 5x^2 - 4x - 12$. Factor $f(x)$ completely, and show all your work. If you use The Remainder theorem, Descartes' Rules of Signs, The Bounds Theorem, or the Rational Root theorem, use complete sentences and explain how you're using them.

5. [6 points] Find the inverse of $\begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix}$ (where $i = \sqrt{-1}$).

6. [4 points] Find all values of k such that $A = \begin{bmatrix} k & -3 & 0 & 3 \\ 9 & 2 & 2 & 5 \\ 0 & 3 & 0 & 0 \\ 12 & 4 & 0 & k \end{bmatrix}$ is invertible.

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7. [5 points] Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & -1 \end{bmatrix}$. Fill in the missing entries of $\mathrm{cof}(A) = \begin{bmatrix} -11 & \boxed{} & -1 \\ 5 & -4 & 1 \\ \boxed{} & \boxed{} & -3 \end{bmatrix}$.

8. [6 points] Calculate: $\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 1+x & x^2 & x^3 \\ 1 & x & 2+x^2 & x^3 \\ 1 & x & x^2 & 3+x^3 \end{vmatrix}$

9. [4 points] A linear system of equations has 4 variables, a, b, c, and d. Find three different solutions for this system, if RREF of the augmented matrix for the system is

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

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10. [10 points] Solve the following system by finding the inverse of the coefficient matrix. **No marks will be awarded for any other method**.

$$\begin{array}{ccccccc} x & + & 3y & + & 2z & = & 5 \\ x & + & 3y & + & 3z & = & 7 \\ & & y & & & = & -2 \end{array}$$

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11. [8 points] Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$. Which of the following expressions make sense? If they don't, explain why.

    (a) $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w})$

    (b) $||\mathbf{v} \bullet \mathbf{u}||$

    (c) $(\mathbf{u} \bullet \mathbf{v}) + \mathbf{u}$

    (d) $(\mathbf{u} \bullet \mathbf{v}) \times \mathbf{w}$

    (e) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

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12. [10 points] Let

$$\mathbf{v}_1 = (3, 6, -9, 3), \qquad \mathbf{v}_2 = (-4, -8, 12, -4), \qquad \mathbf{v}_3 = (1, 1, -2, 1).$$

Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ Linearly Independent or Linearly Dependent? If they are Linearly Independent, prove it. If they are Linearly Dependent, write one of the vectors as a linear combination of the others.

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13. [2 points] Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$T(v_1, v_2, v_3) = (3v_1, 3v_2 - v_3, v_1 + v_2 + 3v_3).$$

Find the matrix A for the linear transformation T.

14. [6 points] Find the intersection of the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$ and the plane $3x - 2y + 4z = 15$.

15. Let T be a linear transformation with associated matrix $A = \begin{bmatrix} 5 & 0 & -3 \\ -4 & 1 & 3 \\ 8 & 0 & -4 \end{bmatrix}$.

(a) [3 points] Find the characteristic equation of T.

(b) [2 points] Given that $\lambda = 1$ is an eigenvalue of A, find the remaining eigenvalues.

(c) [5 points] Find all eigenvectors associated with the eigenvalue $\lambda = 1$.

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16. [0 points] **BONUS: 3 MARKS** (bonus marks only given for substantial progress). Let A be a matrix, $\mathbf{b}$ a column matrix. Let $\mathbf{x}_0$ and $\mathbf{x}_1$ both be solutions to the system of equations $A\mathbf{x} = \mathbf{0}$. Prove that $\mathbf{x}_0 + \mathbf{x}_1$ is a solution to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ as well.

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